§10
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A10.1

(1)

$$K_n := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x \in \left[-1+\frac{1}{n}, 1\right], y \in \left[-\sqrt{1-x^2}, \sqrt{1-x^2}\right], n \in \mathbb{N} \right\}$$

$$\begin{aligned}
\iint_{\mathrm{D}} \frac{1}{\sqrt{1+x}} \mathrm{d}x\mathrm{d}y &= \int_{-1+\frac{1}{n}}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\sqrt{1+x}} \mathrm{d}y\mathrm{d}x \\
&= 2\int_{-1+\frac{1}{n}}^{1} \sqrt{1-x}\mathrm{d}x \\
&= -\frac{4}{3}\left[(1-x)^{\frac{3}{2}}\right]_{-1+\frac{1}{n}}^{1} \\
&= -\frac{4}{3}\left(0-\left(2-\frac{1}{n}\right)^{\frac{3}{2}}\right) \\
&= \lim_{n\to\infty} \frac{4}{3}\left(2-\frac{1}{n}\right)^{\frac{3}{2}} \\
&= \frac{4}{3}\left(\sqrt{2}\right)^3 \\
&< \infty
\end{aligned}$$

$\Longrightarrow$ 収束

(2)

$$K_n := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x \in \left[\frac{1}{n}, \pi\right], y \in [x, \pi], n \in \mathbb{N} \right\}$$

$$\begin{aligned}
\iint_{\mathrm{D}} \frac{\sin x}{y^2} \mathrm{d}x\mathrm{d}y &= \int_{\frac{1}{n}}^{\pi} \int_{x}^{\pi} \frac{\sin x}{y^2} \mathrm{d}y\mathrm{d}x \\
&= \int_{\frac{1}{n}}^{\pi} \sin x \left(\left[-\frac{1}{y}\right]_x^{\pi}\right) \mathrm{d}x \\
&= \int_{\frac{1}{n}}^{\pi} \sin x \left(-\frac{1}{\pi}+\frac{1}{x}\right) \mathrm{d}x \\
&= \int_{\frac{1}{n}}^{\pi} \frac{\sin x}{x}\mathrm{d}x - \frac{1}{\pi}\int_{\frac{1}{n}}^{\pi} \sin x\mathrm{d}x \\
&= \lim_{n\to\infty}\left(\mathrm{Si}(\pi) + \frac{1}{\pi}\left[\cos x\right]_{\frac{1}{n}}^{\pi}\right) \\
&= \mathrm{Si}(\pi) - \frac{2}{\pi} \\
&< \infty
\end{aligned}$$

$\Longrightarrow$ 収束

(3)

$$K_n := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x \in \left[\frac{1}{a}, 1\right], y \in \left[\frac{1}{b}, 1\right], a, b \in \mathbb{N} \right\}$$

$$\begin{aligned}
\iint_{\mathrm{D}} \left(\frac{1}{x} - \frac{1}{y}\right) \mathrm{d}x\mathrm{d}y &= \int_{\frac{1}{a}}^{1} \int_{\frac{1}{b}}^{1} \left(\frac{1}{x} - \frac{1}{y}\right) \mathrm{d}y\mathrm{d}x \\
&= \int_{\frac{1}{a}}^{1} \left(\left[\frac{y}{x} - \log y\right]_{\frac{1}{b}}^{1} \right) \mathrm{d}x \\
&= \int_{\frac{1}{a}}^{1} \left(\frac{1}{x} - \frac{1}{bx} - \log b\right) \mathrm{d}x \\
&= \lim_{\substack{a \to \infty \\ b \to \infty}} \left[\left(1 - \frac{1}{b}\right) \log x - x \log b \right]_{\frac{1}{a}}^{1} \\
&= \lim_{\substack{a \to \infty \\ b \to \infty}} \left(-\log b + \left(1 - \frac{1}{b}\right) \log a + \frac{1}{a} \log b \right) \\
&= \lim_{\substack{a \to \infty \\ b \to \infty}} \left(\left(1 - \frac{1}{b}\right) \log a + \left(\frac{1}{a} - 1\right) \log b \right) \\
&= 0 < \infty
\end{aligned}$$

最後は $a = b = k$ の方向で無限へ近づいた結果である
よって、収束である

A10.2

(1)

$$K_n := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x \in \left[\frac{1}{a}, 1\right], y \in \left[\frac{1}{b}, 1\right], a, b \in \mathbb{N} \right\}$$

$$\begin{aligned}
\iint_{\mathrm{D}} \frac{1}{(x^2 + y^2)^{\frac{3}{2}}} \mathrm{d}x\mathrm{d}y &= \int_{\frac{1}{a}}^{1} \int_{\frac{1}{b}}^{1} \frac{1}{(x^2 + y^2)^{\frac{3}{2}}} \mathrm{d}y\mathrm{d}x \\
&= \int_{\frac{1}{a}}^{1} \left(\left[\frac{y}{x^2 \sqrt{x^2 + y^2}} \right]_{\frac{1}{b}}^{1} \right) \mathrm{d}x \\
&= \int_{\frac{1}{a}}^{1} \left(\frac{1}{x^2 \sqrt{x^2 + 1}} - \frac{1}{x^2 \sqrt{b^2 x^2 + 1}} \right) \mathrm{d}x \\
&= \lim_{\substack{a \to \infty \\ b \to \infty}} \left(\left[\frac{\sqrt{b^2 x^2 + 1}}{x} - \frac{\sqrt{x^2 + 1}}{x} \right]_{\frac{1}{a}}^{1} \right) \\
&= \lim_{\substack{a \to \infty \\ b \to \infty}} \left(\sqrt{b^2 + 1} - \sqrt{2} - \sqrt{a^2 + b^2} + \sqrt{a^2 + 1} \right) \\
&= \infty
\end{aligned}$$

(2)

$$K_n := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x \in \left[0, 1 - \frac{1}{n}\right], y \in [0, x], n \in \mathbb{N} \right\}$$

$$\begin{aligned}
\iint_{\mathrm{D}} \frac{1}{\sqrt{1-xy}} \mathrm{d}x\mathrm{d}y &= \int_0^{1-\frac{1}{n}} \int_0^x \frac{1}{\sqrt{1-xy}} \mathrm{d}y\mathrm{d}x \\
&= \int_0^{1-\frac{1}{n}} \left(\left[-\frac{2\sqrt{1-xy}}{x} \right]_0^x \right) \mathrm{d}x \\
&= \int_0^{1-\frac{1}{n}} \left(-\frac{2\sqrt{1-x^2}}{x} + \frac{2}{x} \right) \mathrm{d}x \\
&= \lim_{n\to\infty} \left(\left[2\log\left(\sqrt{1-x^2}+1\right) - 2\sqrt{1-x^2} \right]_0^{1-\frac{1}{n}} \right) \\
&= \lim_{n\to\infty} 2 \left(\log\left(\sqrt{\frac{2}{n} - \frac{1}{n^2}} + 1 \right) - \sqrt{\frac{2}{n} - \frac{1}{n^2}} - \log 2 + 1 \right) \\
&= 2 - 2\log 2
\end{aligned}$$

(3)

$$K_n := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x \in \left[0, 1 - \frac{1}{n}\right], y \in \left[x^2 + \frac{1}{n}, 1\right], n \in \mathbb{N} \right\}$$

$$\begin{aligned}
\iint_{\mathrm{D}} \frac{1}{\sqrt{y-x^2}} \mathrm{d}x\mathrm{d}y &= \int_0^{1-\frac{1}{n}} \int_{x^2+\frac{1}{n}}^1 \frac{1}{\sqrt{y-x^2}} \mathrm{d}y\mathrm{d}x \\
&= 2\int_0^{1-\frac{1}{n}} \left(\sqrt{1-x^2} - \frac{1}{\sqrt{n}} \right) \mathrm{d}x \\
&= \lim_{n\to\infty} \left(\left[\arctan\left(\frac{x}{\sqrt{1-x^2}} \right) + x\sqrt{1-x^2} - \frac{2}{\sqrt{n}} x \right]_0^{1-\frac{1}{n}} \right) \\
&= \lim_{n\to\infty} \arctan \frac{1-\frac{1}{n}}{\sqrt{\frac{2}{n} - \frac{1}{n^2}}} \\
&= \frac{\pi}{2}
\end{aligned}$$

(4)

$$K_n := \left\{ \left(\begin{array}{c} x \\ y \end{array} \right) \in \mathbb{R}^2 \middle| x \in \left[0, 1 - \frac{1}{n}\right], y \in \left[0, 1 - \frac{1}{n} - x\right], n \in \mathbb{N} \right\}$$

$$\begin{aligned}
\iint_{\mathrm{D}} \frac{1}{\sqrt{1-x-y}} \mathrm{d}x\mathrm{d}y &= \int_0^{1-\frac{1}{n}} \int_0^{1-\frac{1}{n}-x} \frac{1}{\sqrt{1-x-y}} \mathrm{d}y\mathrm{d}x \\
&= \int_0^{1-\frac{1}{n}} \left(\left[-2\sqrt{1-x-y} \right]_0^{1-\frac{1}{n}-x} \right) \mathrm{d}x \\
&= 2\int_0^{1-\frac{1}{n}} \left(\sqrt{1-x} - \frac{1}{\sqrt{n}} \right) \mathrm{d}x \\
&= \lim_{n\to\infty} 2 \left[-\frac{2}{3} (1-x)^{\frac{3}{2}} - \frac{1}{\sqrt{n}} x \right]_0^{1-\frac{1}{n}} \\
&= \lim_{n\to\infty} 2 \left(\frac{1}{3} \left(\frac{1}{n} \right)^{\frac{3}{2}} - \frac{1}{\sqrt{n}} + \frac{2}{3} \right) \\
&= \frac{4}{3}
\end{aligned}$$

A10.3

(1)

$$k_n := \left\{ (r, \theta) \in \mathbb{R}^2 \middle| r \in \left[\frac{1}{n}, 1\right], \theta \in [0, 2\pi], n \in \mathbb{N} \right\}$$

$$\begin{aligned}
\iint_{\mathrm{D}} \frac{1}{(x^2+y^2)^{\frac{\alpha}{2}}} \mathrm{d}x\mathrm{d}y &= \int_0^{2\pi} \int_{\frac{1}{n}}^1 r^{1-\alpha} \mathrm{d}r\mathrm{d}\theta \\
&= \frac{1}{2-\alpha} \int_0^{2\pi} \left(1 - \left(\frac{1}{n} \right)^{2-\alpha} \right) \mathrm{d}\theta \\
&= \lim_{n\to\infty} \frac{2\pi}{2-\alpha} \left(1 - \left(\frac{1}{n} \right)^{2-\alpha} \right) \\
&= \frac{2\pi}{2-\alpha} < \infty
\end{aligned}$$

$\Longrightarrow \alpha \neq 2$ なら、収束し、$\alpha = 2$、発散する

(2)

$$K_n := \left\{ (r, \theta) \in \mathbb{R}^2 \middle| r \in [1, n), \theta \in [0, 2\pi], n \in \mathbb{N} \right\}$$

$$\begin{aligned}
\iint_{\mathrm{D}} \frac{1}{(x^2+y^2)^{\frac{\alpha}{2}}} \mathrm{d}x\mathrm{d}y &= \int_0^{2\pi} \int_1^n r^{1-\alpha} \mathrm{d}r\mathrm{d}\theta \\
&= \frac{1}{2-\alpha} \int_0^{2\pi} \left(n^{2-\alpha} - 1 \right) \mathrm{d}\theta \\
&= \lim_{n\to\infty} \frac{2\pi}{2-\alpha} \left(n^{2-\alpha} - 1 \right)
\end{aligned}$$

$\Longrightarrow \alpha > 2$、収束し、$\alpha \leq 2$、発散する

(3)

$$K_n := \left\{ \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} \in \mathbb{R}^3 \middle| r \in \left[\frac{1}{n}, 1\right], \theta \in [0, 2\pi], \phi \in [0, \pi] \right\}$$

$$\begin{aligned}
\iiint_{\mathrm{D}} \frac{1}{(x^2+y^2+z^2)^{\frac{\alpha}{2}}} \mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_{\frac{1}{n}}^{1} \int_0^{2\pi} \int_0^{\pi} r^{2-\alpha} \sin\phi \mathrm{d}\phi \mathrm{d}\theta \mathrm{d}r \\
&= \int_{\frac{1}{n}}^{1} \int_0^{2\pi} r^{2-\alpha} \left[-\cos\phi\right]_0^{\pi} \mathrm{d}\theta \mathrm{d}r \\
&= 4\pi \int_{\frac{1}{n}}^{1} r^{2-\alpha} \mathrm{d}r \\
&= \frac{4\pi}{3-\alpha} \left(1 - \left(\frac{1}{n}\right)^{3-\alpha}\right)
\end{aligned}$$

$\Longrightarrow \alpha \neq 3$、収束し、$\alpha = 3$、発散する

(4)

$$K_n := \left\{ \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} \in \mathbb{R}^3 \middle| r \in [1, n], \theta \in [0, 2\pi], \phi \in [0, \pi] \right\}$$

$$\begin{aligned}
\iiint_{\mathrm{D}} \frac{1}{(x^2+y^2+z^2)^{\frac{\alpha}{2}}} \mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_1^{n} \int_0^{2\pi} \int_0^{\pi} r^{2-\alpha} \sin\phi \mathrm{d}\phi \mathrm{d}\theta \mathrm{d}r \\
&= 4\pi \int_1^{n} r^{2-\alpha} \mathrm{d}r \\
&= \frac{4\pi}{3-\alpha} \left(n^{3-\alpha} - 1\right)
\end{aligned}$$

$\Longrightarrow \alpha > 3$、収束し、$\alpha \leq 3$、発散する

B10.4

(1)

$$k_n := \left\{ (r, \theta) \in \mathbb{R}^2 \middle| r \in \left[\frac{1}{n}, 1\right], \theta \in [0, 2\pi], n \in \mathbb{N} \right\}$$

$$\begin{aligned}
\iint_{\mathrm{D}} \frac{1}{\log\left(1 + (x^2+y^2)^{\frac{\alpha}{2}}\right)} \mathrm{d}x\mathrm{d}y &= \int_{\frac{1}{n}}^{1} \int_0^{2\pi} \frac{r}{\log(1+r^{\alpha})} \mathrm{d}\theta \mathrm{d}r \\
&= 2\pi \int_{\frac{1}{n}}^{1} \frac{r}{\log(1+r^{\alpha})} \mathrm{d}r
\end{aligned}$$

ここはもう答えは特殊関数であることがわかるから直接に計算する必要はない、以下ははさみうちで有界であることを証明すれば収束であることがわかれる

そして $0 < \dfrac{r}{\log(1+r^{\alpha})} \leq \dfrac{1}{\log(1+r^{\alpha})}$

言い換えれば、上界さえあればいい

$\alpha \leq 0$ では $\dfrac{r}{\log(1+r^\alpha)} \leq \dfrac{1}{\log(1+r^\alpha)} \leq \dfrac{1}{\log 2}$ から、明らかに有界である
$\alpha \in [0,1]$ では $\dfrac{r}{\log(1+r^\alpha)} \leq \dfrac{1}{\log 2}$ もあるから、この範囲でも有界である
$\alpha > 1$ の場合は、$\alpha \longrightarrow \infty$ とともに、$\dfrac{r}{\log(1+r^\alpha)} \longrightarrow \infty$ から、限界はできるだけ小さいほうがいい
ここで注意力を使って、$\alpha = 2$ のとき、$\displaystyle\int_{\frac{1}{n}}^{1} \frac{r}{\log(1+r^\alpha)} \mathrm{d}r = -\frac{1}{2}\mathrm{Li}\left(1+\frac{1}{n^2}\right)$
さらに $n \longrightarrow \infty \Longrightarrow \dfrac{1}{2}\mathrm{Li}\left(1+\dfrac{1}{1+n^2}\right) \longrightarrow \infty$ から、$\alpha = 2$ がとれない
だから $\alpha < 2$ なら有界である
以上より、$\alpha < 2$ とすれば収束する

(2)

$K_n := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x \in \left[\frac{1}{n}, \pi\right], y \in [x, \pi], n \in \mathbb{N} \right\}$

$$\begin{aligned}
\iint_{\mathrm{D}} \frac{\sin x}{y^\alpha} \mathrm{d}x\mathrm{d}y &= \int_{\frac{1}{n}}^{\pi} \int_{x}^{\pi} \frac{\sin x}{y^\alpha} \mathrm{d}y\mathrm{d}x \\
&= \begin{cases} \displaystyle\int_{\frac{1}{n}}^{\pi} (\pi - x)\sin x \mathrm{d}x & \alpha = 0 \\ \displaystyle\int_{\frac{1}{n}}^{\pi} (\log\pi - \log x)\sin x \mathrm{d}x & \alpha = 1 \\ \displaystyle\frac{1}{1-\alpha}\int_{\frac{1}{n}}^{\pi} \left(\pi^{1-\alpha} - x^{1-\alpha}\right)\sin x \mathrm{d}x & \alpha \neq 0, a \neq 1 \end{cases} \\
&\overset{n\to\infty}{=} \begin{cases} \pi & \alpha = 0 \\ \gamma - \mathrm{Ci}(\pi) + \log\pi & \alpha = 1 \\ \dfrac{2}{1-\alpha}\pi^{1-\alpha} - \dfrac{1}{1-\alpha} I & \alpha \neq 0, 1 \end{cases}
\end{aligned}$$

ここで $I = \displaystyle\int_{\frac{1}{n}}^{\pi} x^{1-\alpha} \sin x \mathrm{d}x$

$$\begin{aligned}
I &= \left[-x^{1-\alpha}\cos x\right]_{\frac{1}{n}}^{\pi} + \frac{1}{1-\alpha}\int_{\frac{1}{n}}^{\pi} x^{-\alpha}\cos x \mathrm{d}x \\
&= \pi^{1-\alpha} + \left(\frac{1}{n}\right)^{1-\alpha}\cos\frac{1}{n} + \frac{1}{1-\alpha}\int_{\frac{1}{n}}^{\pi} x^{-\alpha}\cos x \mathrm{d}x \\
&\overset{n\to\infty}{=} \pi^{1-\alpha} - \left[-\pi^{1-\alpha}\cos x + 2i(-ix)^\alpha x^{-\alpha}\Gamma(2-\alpha, -ix) - 2i(ix)^\alpha x^{-\alpha}\Gamma(2-\alpha, ix)\right]_{\frac{1}{n}}^{\pi}
\end{aligned}$$

これになると有界性が判断しにくいから、一応上界関数を考える
$\alpha \leq 0$ では、$0 \leq \displaystyle\int_{\frac{1}{n}}^{\pi} \left(\pi^{1-\alpha} - x^{1-\alpha}\right)\sin x \mathrm{d}x \leq \int_{\frac{1}{n}}^{\pi} \pi^{1-\alpha}\sin x \mathrm{d}x$
めでたいことは、右側の積分が計算しやすくて、$n \longrightarrow \infty$ なら結果が $2\pi^{1-\alpha}$、有界
$0 \leq \alpha \leq 1$ なら、$\left(\pi^{1-\alpha} - x^{1-\alpha}\right)\sin x$ の明らかに上界がある（例えば 2）
$1 \leq \alpha \leq 2$ なら、上界ではなく、下界を考えるべきだから、明らかに -2 は下界である（なぜなら $\sin x$ がこの区間内で有界である）$\alpha = 2$ はもう A10.1.2 で計算したから、有界である
$\alpha > 2$ なら、累次積分の中の積分結果から出た式は $\dfrac{\sin x}{x^{\alpha-1}}$
特異点は 0 だからこのあたりを考えるから、$x \to 0 \Longrightarrow x^{\alpha-1} \overset{faster}{\longrightarrow} 0$

言い換えれば $\dfrac{\sin x}{x^{\alpha-1}} \longrightarrow \infty$
だから、$\alpha \leq 2$ の場合は収束

(3)

α, β のどちら一方が $-$ になると範囲内に明らかに有界であるから、以下は両方がともに $+$ の場合を考えよう
対称性を考えると、x, y がともに $+$ だけ考えればいい
が、α, β が 0 以上になると、断面は反比例関数だから、$x\ or\ y \longrightarrow 0 \Longrightarrow$ 積分結果が無限になる
よって、$\alpha \leq 0, \beta \leq 0$

(4)

$$K_n := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x \in [1, n], y \in [1, x], n \in \mathbb{N} \right\}$$

$$\begin{aligned}
\iint_{\mathrm{D}} \frac{1}{x^\alpha y^\beta} \mathrm{d}x\mathrm{d}y &= \int_1^n \int_1^x \frac{1}{x^\alpha y^\beta} \mathrm{d}y\mathrm{d}x \\
&= \begin{cases} \displaystyle\int_1^n \frac{1}{x^\alpha}(x-1)\,\mathrm{d}x = I_1 & \beta = 0 \\ \displaystyle\int_1^n \frac{1}{x^\alpha} \log x \mathrm{d}x = I_2 & \beta = 1 \\ \displaystyle\frac{1}{1-\beta}\int_1^n \frac{1}{x^\alpha}\left(x^{1-\beta} - 1\right)\mathrm{d}x = I_3 & \beta \neq 0, 1 \end{cases}
\end{aligned}$$

$\beta = 0$

$$\begin{aligned}
I_1 &= \begin{cases} \displaystyle\int_1^n (x-1)\,\mathrm{d}x & \alpha = 0 \\ \displaystyle\int_1^n \left(1 - \frac{1}{x}\right)\mathrm{d}x & \alpha = 1 \\ \displaystyle\int_1^n \left(\frac{1}{x^{\alpha-1}} - \frac{1}{x^\alpha}\right)\mathrm{d}x & \alpha \neq 0, 1 \end{cases} \\
&= \begin{cases} \dfrac{1}{2}n^2 - n + \dfrac{1}{2} & \alpha = 0 \\ n - \log n - 1 & \alpha = 1 \\ \dfrac{1}{2-\alpha}\left(\dfrac{1}{n^{\alpha-2}} - 1\right) - \dfrac{1}{1-\alpha}\left(1 - \dfrac{1}{n^{\alpha-1}}\right) & \alpha \neq 0, 1 \end{cases} \\
&\overset{n \to \infty}{=} \begin{cases} \infty & \alpha = 0 \\ \infty & \alpha = 1 \\ \infty & \alpha = 2 \\ \dfrac{1}{\alpha-2} - \dfrac{1}{1-\alpha} & \alpha \neq 0, 1, 2 \end{cases}
\end{aligned}$$

$\beta = 1$

$$
\begin{aligned}
I_2 &= \begin{cases} \displaystyle\int_1^n \log x \mathrm{d}x & \alpha = 0 \\ \displaystyle\int_1^n \frac{\log x}{x} \mathrm{d}x & \alpha = 1 \\ \displaystyle\int_1^n \frac{\log x}{x^\alpha} \mathrm{d}x & \alpha \neq 0, 1 \end{cases} \\
&= \begin{cases} n \log n - n + 1 & \alpha = 0 \\ \dfrac{1}{2} (\log n)^2 & \alpha = 1 \\ \dfrac{1}{(\alpha - 1)^2} - \dfrac{(\alpha - 1) \log n + 1}{(\alpha - 1)^2} n^{1-\alpha} & \alpha \neq 0, 1 \end{cases} \\
&\overset{n \to \infty}{=} \begin{cases} \infty & \alpha = 0 \\ \infty & \alpha = 1 \\ -\infty & \alpha < 0 \\ \dfrac{1}{(\alpha - 1)^2} & \alpha \in (0, 1) \cup (1, \infty) \end{cases}
\end{aligned}
$$

$\beta \neq 0, 1$

$$
\begin{aligned}
I_3 &= \frac{1}{1 - \beta} \int_1^n \left(\frac{1}{x^{\alpha + \beta - 1}} - \frac{1}{x^\alpha} \right) \mathrm{d}x \\
&= \begin{cases} \dfrac{1}{1-\beta} \left(n - \dfrac{1}{1-\alpha} n^{1-\alpha} + \dfrac{\alpha}{1-\alpha} \right) & \alpha + \beta = 1 \\ \dfrac{1}{1-\beta} (\log n - n + 1) & \alpha + \beta = 2, \alpha = 0 \\ \dfrac{1}{1-\beta} \left(\log n - \dfrac{1}{1-\alpha} n^{1-\alpha} + \dfrac{1}{1-\alpha} \right) & \alpha + \beta = 2, \alpha \neq 0 \\ \dfrac{1}{1-\beta} \left(\dfrac{1}{2-\alpha-\beta} n^{2-\alpha-\beta} - \log n - \dfrac{1}{2-\alpha-\beta} \right) & \alpha + \beta \neq 2, \alpha = 1 \\ \dfrac{1}{1-\beta} \left(\dfrac{1}{2-\alpha-\beta} n^{2-\alpha-\beta} - \dfrac{1}{1-\alpha} n^{1-\alpha} - \dfrac{1}{2-\alpha-\beta} + \dfrac{1}{1-\alpha} \right) & \alpha + \beta \neq 2, \alpha \neq 1 \end{cases} \\
&= \begin{cases} \infty & \alpha + \beta = 1 \\ \pm\infty & \alpha + \beta = 2, \alpha = 0 \\ \pm\infty & \alpha + \beta = 2, \alpha \neq 0 \\ \pm\infty & \alpha + \beta \leq 2, \alpha \leq 1 \\ \dfrac{1}{1-\beta} \left(\dfrac{1}{1-\alpha} - \dfrac{1}{2-\alpha-\beta} \right) & \alpha + \beta > 2, \alpha > 1 \end{cases}
\end{aligned}
$$

以上の醜い計算より、収束する範囲は
$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in [(\mathbb{R} \setminus \{0, 1, 2\}) \times \{0\}] \bigcup [(\mathbb{R}^+ \setminus \{1\}) \times \{1\}] \bigcup \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^2 \middle| \alpha + \beta > 2, \alpha > 1, \beta \neq 0, 1 \right\}$

B10.5

(1)

$$\begin{aligned}I_1 &= \lim_{\epsilon\to 0}\int_{\epsilon}^{1}\frac{1}{(x+1)^2}\mathrm{d}x\\&= \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon+1}-\frac{1}{2}\right)\\&= \frac{1}{2}\end{aligned}$$

(2)

$$\begin{aligned}I_2 &= \lim_{\epsilon\to 0}\int_{\epsilon}^{1}\left(-\frac{1}{(y+1)^2}\right)\mathrm{d}y\\&= \lim_{\epsilon\to 0}\left(\frac{1}{2}-\frac{1}{\epsilon+1}\right)\\&= -\frac{1}{2}\end{aligned}$$

(3)

対称性を考えると、$\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \middle| x \in [-1,1], y \in [0,1]\right\}$ で積分結果を二倍すればいい

$$\begin{aligned}I_3 &= \lim_{\epsilon\to 0}\int_{\epsilon-1}^{1}\frac{1}{(x+1)^2}\mathrm{d}x\\&= \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}-\frac{1}{2}\right)\\&= \infty\end{aligned}$$

B10.6

(1)

Thm10.5 より、ここの優関数は $\phi = r$
n 次元の球の体積を考える

$$\begin{aligned}\int_{\mathrm{D}_1} f\mathrm{d}\mathbf{x} &= C\int_0^1 r^{\alpha+n-1}\mathrm{d}r\\&= \begin{cases}\dfrac{1}{\alpha+n} & \alpha+n>0\\ \infty & \alpha+n\le 0\end{cases}\end{aligned}$$

$\Longrightarrow \alpha > -n$ は収束する条件

(2)

(1) と同じように

$$\begin{aligned}\int_{\mathrm{D}_2} f\mathrm{d}\mathbf{x} &= C\int_1^\infty r^{\alpha+n-1}\mathrm{d}r \\ &= \begin{cases} -\dfrac{1}{\alpha+n} & \alpha+n<0 \\ \infty & \alpha+n\geq 0\end{cases}\end{aligned}$$

$\Longrightarrow \alpha < -n$ は収束する条件

B10.7

(1)

$$\begin{aligned}\iint_{\mathrm{D}} \left(x^2y^2\right)^j \mathrm{d}x\mathrm{d}y &= \int_0^1 x^{2j}\left(\int_0^1 y^{2j}\mathrm{d}y\right)\mathrm{d}x \\ &= \frac{1}{2j+1}\int_0^1 x^{2j}\mathrm{d}x \\ &= \frac{1}{(2j+1)^2}\end{aligned}$$

(2)

$$\mathrm{J} := \begin{vmatrix} \dfrac{\cos u}{\cos v} & \dfrac{\sin u\sin v}{\cos^2 v} \\ \dfrac{\sin u\sin v}{\cos^2 u} & \dfrac{\cos v}{\cos u}\end{vmatrix} = 1-\tan^2 u\tan^2 v$$

$$\begin{aligned}\iint_{\mathrm{D}} \frac{1}{1-x^2y^2}\mathrm{d}x\mathrm{d}y &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}-u}\frac{1}{1-\tan^2 u\tan^2 v}\left(1-\tan^2 u\tan^2 v\right)\mathrm{d}v\mathrm{d}u \\ &= \int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-u\right)\mathrm{d}u \\ &= \frac{\pi^2}{8}\end{aligned}$$

(3)

$$\begin{aligned}\mathrm{LHS} &= \sum_{j=1}^{\infty}\frac{1}{(2j+1)^2} \\ &\overset{(1)}{=} \sum_{j=1}^{\infty}\iint_{\mathrm{D}}\left(x^2y^2\right)^j\mathrm{d}x\mathrm{d}y \\ &= \iint_{\mathrm{D}}\left(\sum_{j=1}^{\infty}\left(x^2y^2\right)^j\right)\mathrm{d}x\mathrm{d}y \\ &\overset{(10.8)}{=} \iint_{\mathrm{D}}\frac{1}{1-x^2y^2}\mathrm{d}x\mathrm{d}y \\ &= \frac{\pi^2}{8}\end{aligned}$$

バーセル問題
まず級数 $\zeta(2)$ の積分表示を考えよう

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^2} &= \sum_{n=1}^{\infty}\frac{y^n}{n^2}\Big|_{y=0}^{y=1} \\
&= \int_0^1 \sum_{n=1}^{\infty}\frac{y^{n-1}}{n}\mathrm{d}y \\
&= \int_0^1 \left(\int_0^1 \sum_{n=1}^{\infty}(xy)^{n-1}\,\mathrm{d}x\right)\mathrm{d}y \\
&= \int_0^1\int_0^1 \frac{1}{1-xy}\mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathrm{D}} \frac{1}{1-xy}\mathrm{d}x\mathrm{d}y
\end{aligned}$$

変換 $\begin{pmatrix} x \\ y \end{pmatrix} := \Psi(u,v) = \begin{pmatrix} \frac{u-v}{\sqrt{2}} \\ \frac{u+v}{\sqrt{2}} \end{pmatrix}$ を考えると、範囲 D' は $(0,0)\,,\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),(\sqrt{2},0)\,,\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ からなる正方形である

$$\begin{aligned}
\zeta(2) &= \iint_{\mathrm{D}} \frac{1}{1-xy}\mathrm{d}x\mathrm{d}y \\
&= 2\iint_{\mathrm{D}'} \frac{1}{2-u^2+v^2}\mathrm{d}u\mathrm{d}v \\
&= 4\int_0^{\frac{1}{\sqrt{2}}}\int_0^{u}\frac{1}{2-u^2+v^2}\mathrm{d}v\mathrm{d}u + 4\int_{\frac{1}{\sqrt{2}}}^{\sqrt{2}}\int_0^{\sqrt{2}-u}\frac{1}{2-u^2+v^2}\mathrm{d}v\mathrm{d}u \\
&= 4\int_0^{\frac{\pi}{6}}\theta\mathrm{d}\theta + 4\int_0^{\frac{\pi}{3}}\frac{\theta}{2}\mathrm{d}\theta \\
&= \frac{\pi^2}{18}+\frac{\pi^2}{9} \\
&= \frac{\pi^2}{6}
\end{aligned}$$

参考文献