§12
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A12.1

(1)

$$\begin{aligned}\int_{-\infty}^{\infty} f(t,x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} e^{-tx^2}\mathrm{d}x \\ &< \int_{-\infty}^{\infty} e^{0}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} 1\mathrm{d}x \\ &= \lim_{\substack{L\to-\infty \\ R\to\infty}} \int_{L}^{R} 1\mathrm{d}x \\ &= \infty\end{aligned}$$

言い換えれば、左側は可積分

$$\begin{aligned}\int_{-\infty}^{\infty} \left|\partial_t f(t,x)\right|\mathrm{d}x &= \int_{-\infty}^{\infty} \left|\left(-x^2\right)e^{-tx^2}\right|\mathrm{d}x \\ &< \int_{-\infty}^{\infty} \frac{1}{te}\mathrm{d}x \\ &= \lim_{\substack{L\to-\infty \\ R\to\infty}} \int_{L}^{R} \frac{1}{te}\mathrm{d}x \\ &= \infty\end{aligned}$$

右側も可積分
よって、定理 12.2 より等しい
また、この定理の証明について $G(\alpha) := \int_a^b f_\alpha(x,\alpha)\,\mathrm{d}x$

$$\begin{aligned}\int_{\alpha_1}^{\alpha} G(\alpha)\,\mathrm{d}\alpha &= \int_{\alpha_1}^{\alpha}\mathrm{d}\alpha \int_a^b f_\alpha(x,\alpha)\,\mathrm{d}x \\ &= \int_a^b \mathrm{d}x \int_{\alpha_1}^{\alpha} f_\alpha(x,\alpha)\,\mathrm{d}\alpha \\ &= \int_a^b \left(f(x,\alpha) - f(x,\alpha_1)\right)\mathrm{d}x \\ &= F(\alpha) - F(\alpha_1)\end{aligned}$$

両辺を微分すれば得られる

(2)

$$\begin{aligned}\int_{-\infty}^{\infty} x^2 e^{-tx^2}\mathrm{d}x &= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} e^{-tx^2}\mathrm{d}x \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} e^{-t\cdot\frac{u^2}{t}}\frac{1}{\sqrt{t}}\mathrm{d}u \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-u^2}\mathrm{d}u \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\sqrt{\pi}}{\sqrt{t}} \\ &= \frac{\sqrt{\pi}}{2}t^{-\frac{3}{2}}\end{aligned}$$

A12.2

(1)

$$\begin{aligned}I\left(0\right) &= \int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x \\ &= \sqrt{\pi}\end{aligned}$$

(2)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a}I\left(a\right) &= \frac{\mathrm{d}}{\mathrm{d}a}\int_{-\infty}^{\infty} e^{-x^2}\cos\left(2ax\right)\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\partial_a\left(e^{-x^2}\cos\left(2ax\right)\right)\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\left(-2xe^{-x^2}\sin\left(2ax\right)\right)\mathrm{d}x \\ &= -\int_{-\infty}^{\infty} e^{-u}\sin\left(2a\sqrt{u}\right)\mathrm{d}u \\ &= -\frac{1}{2a^2}\int_{-\infty}^{\infty} ve^{-\frac{v^2}{4a^2}}\sin v\mathrm{d}v \\ &= -\frac{1}{2a^2}\left(\left[-2a^2e^{-\frac{v^2}{4a^2}}\sin v\right]_{-\infty}^{\infty} + 2a^2\int_{-\infty}^{\infty} e^{-\frac{v^2}{4a^2}}\cos v\mathrm{d}v\right) \\ &= 0 - \int_{-\infty}^{\infty} e^{-\frac{v^2}{4a^2}}\cos v\mathrm{d}v \\ &= -I\left(a\right)\end{aligned}$$

A12.3

(1)

$$\begin{aligned}
I &= \int_0^\infty e^{-\alpha x} \sin \beta x \mathrm{d}x \\
&= \left[-\frac{1}{\alpha} e^{-\alpha x} \sin \beta x \right]_0^\infty + \frac{\beta}{\alpha} \int_0^\infty e^{-\alpha x} \cos \beta x \mathrm{d}x \\
&= \frac{\beta}{\alpha} \int_0^\infty e^{-\alpha x} \cos \beta x \mathrm{d}x \\
&= \frac{\beta}{\alpha} \left(\left[-\frac{1}{\alpha} e^{-\alpha x} \cos \beta x \right]_0^\infty - \frac{\beta}{\alpha} \int_0^\infty e^{-\alpha x} \sin \beta x \mathrm{d}x \right) \\
&= \frac{\beta}{\alpha} \left(\frac{1}{\alpha} - \frac{\beta}{\alpha} I \right)
\end{aligned}$$

$\Longrightarrow I = \dfrac{\beta}{\alpha^2 + \beta^2}$

(2)

$$\int_0^\infty \int_0^\infty e^{-xy} \sin x \mathrm{d}y \mathrm{d}x = \int_0^\infty \frac{\sin x}{x} \mathrm{d}x$$

$$\int_0^\infty \int_0^\infty e^{-xy} \sin x \mathrm{d}x \mathrm{d}y \overset{\alpha = y, \beta = 1}{=} \int_0^\infty \frac{1}{y^2 + 1} \mathrm{d}y$$

以上より、順序交換できない

(3)

$$\begin{aligned}
\text{RHS} &= \int_0^L \left(\int_0^R e^{-xy} \sin x \mathrm{d}x \right) \mathrm{d}y + \int_0^R \frac{\sin x}{x} e^{-Lx} \mathrm{d}x \\
&= \int_0^R \left(\int_0^L e^{-xy} \sin x \mathrm{d}y \right) \mathrm{d}x + \int_0^R \frac{\sin x}{x} e^{-Lx} \mathrm{d}x \\
&= \int_0^R \left(\sin x \left(-\frac{1}{L} e^{-Lx} + \frac{1}{x} \right) \right) \mathrm{d}x + \int_0^R \frac{\sin x}{x} e^{-Lx} \mathrm{d}x \\
&= \int_0^R \frac{\sin x}{x} \mathrm{d}x - \frac{1}{L} \int_0^R e^{-Lx} \sin x \mathrm{d}x + \int_0^R \frac{\sin x}{x} e^{-Lx} \mathrm{d}x \\
&= \int_0^R \frac{\sin x}{x} \mathrm{d}x \\
&= \text{LHS}
\end{aligned}$$

二つ目の式は (2) でもう証明したから略

(4)

$$\begin{aligned}\int_0^R \frac{\sin x}{x}\mathrm{d}x &= \int_0^\infty \int_0^R e^{-xy}\sin x \mathrm{d}x\mathrm{d}y \\ &= \int_0^\infty \frac{e^{-Ry}\left(e^{Ry}-\cos R - y\sin R\right)}{y^2+1}\mathrm{d}y \\ &= \int_0^\infty \frac{1}{y^2+1}\mathrm{d}y - \int_0^\infty \frac{e^{-Ry}}{y^2+1}\left(\cos R + y\sin R\right)\mathrm{d}y \\ &= \frac{\pi}{2} - \int_0^\infty \frac{e^{-Ry}}{y^2+1}\left(\cos R + y\sin R\right)\mathrm{d}y\end{aligned}$$

(5)

$$\begin{aligned}\int_0^\infty \left|\frac{e^{-Ry}}{y^2+1}\left(\cos R + y\sin R\right)\right|\mathrm{d}y &= \int_0^\infty \frac{e^{-Ry}}{y^2+1}\left|\cos R + y\sin R\right|\mathrm{d}y \\ &\leq \int_0^\infty \frac{e^{-Ry}}{y^2+1}\cdot\sqrt{1+y^2}\mathrm{d}y \\ &= \int_0^\infty \frac{e^{-Ry}}{\sqrt{y^2+1}}\mathrm{d}y \\ &< \int_0^\infty e^{-Ry}\mathrm{d}y \\ &\xrightarrow{n\to\infty} 0\end{aligned}$$

また、絶対値よりこの積分は 0 より大きいから、はさみうち原理より 0 に収束

B12.4

(1)

$$\begin{aligned}\int_{\mathbb{R}^N} K\left(\mathbf{x}-\mathbf{y}\right)\phi\left(\mathbf{y}\right)\mathrm{d}\mathbf{y} &= \int_{\mathbb{R}^N} \frac{1}{\left(2-N\right)\omega_N}\left\|\mathbf{x}-\mathbf{y}\right\|^{2-N}\phi\left(y\right)\mathrm{d}y \\ &= \int_{B(x,\epsilon)} \frac{1}{\left(2-N\right)\omega_N}\left\|\mathbf{x}-\mathbf{y}\right\|^{2-N}\phi\left(y\right)\mathrm{d}y \\ &+ \int_{\mathbb{R}^N\setminus B(x,\epsilon)} \frac{1}{\left(2-N\right)\omega_N}\left\|\mathbf{x}-\mathbf{y}\right\|^{2-N}\phi\left(y\right)\mathrm{d}y \\ &= \int_{B(x,\epsilon)} \frac{1}{\left(2-N\right)\omega_N}\left\|\mathbf{x}-\mathbf{y}\right\|^{2-N}\phi\left(y\right)\mathrm{d}y \quad y\notin K, \phi\left(y\right)=0 \\ &= \int_0^\epsilon \int_{S^{N-1}} \frac{1}{\left(2-N\right)\omega_N} r^{2-N}r^{N-1}\phi\left(y\right)\mathrm{d}\sigma\mathrm{d}r \quad \int_{S^{N-1}}\mathrm{d}\sigma = \omega_N \\ &= \frac{1}{2-N}\int_0^\epsilon r^{1-N}\phi\left(y\right)\mathrm{d}r \\ &\sim \frac{1}{2-N}\int_0^\epsilon r^{1-N}\mathrm{d}r \\ &= \frac{1}{\left(2-N\right)^2}\epsilon^{2-N}\end{aligned}$$

(2)

$$
\begin{aligned}
\Delta_{\mathbf{x}} K(\mathbf{x}-\mathbf{y}) &= \Delta_{\mathbf{x}} \frac{1}{(2-N)\,\omega_N} \|\mathbf{x}-\mathbf{y}\|^{2-N} \\
&= \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} \frac{1}{(2-N)\,\omega_N} \|\mathbf{x}-\mathbf{y}\|^{2-N} \\
&= \frac{1}{(2-N)\,\omega_N} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} \|\mathbf{x}-\mathbf{y}\|^{2-N} \\
&= \frac{1}{(2-N)\,\omega_N} \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left((2-N) \|\mathbf{x}-\mathbf{y}\|^{1-N} \cdot \frac{x_j - y_j}{\|\mathbf{x}-\mathbf{y}\|} \right) \\
&= \frac{1}{(2-N)\,\omega_N} \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left((2-N) \|\mathbf{x}-\mathbf{y}\|^{-N} \cdot (x_j - y_j) \right) \\
&= \frac{1}{(2-N)\,\omega_N} \sum_{j=1}^{N} \left(-N \|\mathbf{x}-\mathbf{y}\|^{-N-2} (x_j - y_j)^2 + N \|\mathbf{x}-\mathbf{y}\|^{-N} \right) \\
&= \frac{1}{(2-N)\,\omega_N} \sum_{j=1}^{N} \left(-N \|\mathbf{x}-\mathbf{y}\|^{-N} + N \|\mathbf{x}-\mathbf{y}\|^{-N} \right) \\
&= 0
\end{aligned}
$$

(3)

$$
\begin{aligned}
\Delta_{\mathbf{x}} u(\mathbf{x}) &= \Delta_{\mathbf{x}} \int_{\mathbb{R}^N} K(\mathbf{x}-\mathbf{y})\,\phi(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&= \int_{\mathbb{R}^N} \Delta_{\mathbf{x}} K(\mathbf{x}-\mathbf{y})\,\phi(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&= \int_{\mathbb{R}^N} 0 \cdot \phi(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&= 0
\end{aligned}
$$

(4)

$\mathbf{x} = \mathbf{y}$ なら、その $\|\mathbf{x}-\mathbf{y}\|^{2-N}$ は無限になるから、その積分も広義積分不可だから、交換できない

個人的な感想：今回の B 問題は ↑ こんな感じです.