A3.1

(1)

$$\begin{aligned}\nabla f(x,y) &= {}^t\left(\partial_x f(x,y), \partial_y f(x,y)\right)\\ &= {}^t\left(3x^2+4xy+3y^2, 2x^2+6xy+12y^2\right)\end{aligned}$$

$f_{xx} = 6x+4y \qquad f_{xy} = 4x+6y \qquad f_{yx} = 4x+6y \qquad f_{yy} = 6x+24y$ で

$$H_f(x,y) = \begin{pmatrix} 6x+4y & 4x+6y \\ 4x+6y & 6x+24y \end{pmatrix}$$

$$\begin{aligned}f(0,b) &= f(1,0) + \sum_{k=1}^{3} \frac{1}{k!}\left(\left(\left({}^t(0,b) - {}^t(1,0)\right)\cdot\nabla\right)^k f\right)\left({}^t(1,0)\right)\\ &= f(1,0) + \sum_{k=1}^{3} \frac{1}{k!}\left(\left(\left({}^t(-1,b)\right)\cdot\nabla\right)^k f\right)\left({}^t(1,0)\right)\\ &= f(1,0) + \sum_{k=1}^{3} \frac{1}{k!}\left(\left(-\partial_x + b\partial_y\right)^k f\right)\left({}^t(1,0)\right)\\ &= \dots\end{aligned}$$

(2)

$$\begin{aligned}\nabla f(x,y) &= {}^t\left(\partial_x f(x,y), \partial_y f(x,y)\right)\\ &= {}^t\left(-\frac{2\pi xy}{(1+x^2)^2}\cos\frac{\pi y}{1+x^2}, \frac{\pi}{1+x^2}\cos\frac{\pi y}{1+x^2}\right)\end{aligned}$$

$$f_{xx} = -\frac{4\pi^2x^2y^2\sin\left(\frac{\pi y}{x^2+1}\right)}{(x^2+1)^4} + \frac{8\pi x^2 y\cos\left(\frac{\pi y}{x^2+1}\right)}{(x^2+1)^3} - \frac{2\pi y\cos\left(\frac{\pi y}{x^2+1}\right)}{(x^2+1)^2}$$

$$f_{xy} = \frac{2\pi^2 xy \sin\left(\frac{\pi y}{x^2+1}\right)}{\left(x^2+1\right)^3} - \frac{2\pi x \cos\left(\frac{\pi y}{x^2+1}\right)}{\left(x^2+1\right)^2}$$

$$f_{yx} = \frac{2\pi^2 xy \sin\left(\frac{\pi y}{x^2+1}\right)}{\left(x^2+1\right)^3} - \frac{2\pi x \cos\left(\frac{\pi y}{x^2+1}\right)}{\left(x^2+1\right)^2}$$

$$f_{yy} = -\frac{\pi^2 \sin\left(\frac{\pi y}{x^2+1}\right)}{\left(x^2+1\right)^2}$$

$$H_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

ここで、

$$f_{xx} = -\frac{4\pi^2 x^2 y^2 \sin\left(\frac{\pi y}{x^2+1}\right)}{\left(x^2+1\right)^4} + \frac{8\pi x^2 y \cos\left(\frac{\pi y}{x^2+1}\right)}{\left(x^2+1\right)^3} - \frac{2\pi y \cos\left(\frac{\pi y}{x^2+1}\right)}{\left(x^2+1\right)^2}$$

$$f_{xy} = \frac{2\pi^2 xy \sin\left(\frac{\pi y}{x^2+1}\right)}{\left(x^2+1\right)^3} - \frac{2\pi x \cos\left(\frac{\pi y}{x^2+1}\right)}{\left(x^2+1\right)^2}$$

$$f_{yx} = \frac{2\pi^2 xy \sin\left(\frac{\pi y}{x^2+1}\right)}{\left(x^2+1\right)^3} - \frac{2\pi x \cos\left(\frac{\pi y}{x^2+1}\right)}{\left(x^2+1\right)^2}$$

$$f_{yy} = -\frac{\pi^2 \sin\left(\frac{\pi y}{x^2+1}\right)}{\left(x^2+1\right)^2}$$

(3)

$$\begin{aligned} \nabla f(x,y) &=^t \left(\partial_x f(x,y), \partial_y f(x,y)\right) \\ &=^t \left(-2e^{-\left(x^2+y^2\right)}x, -2e^{-\left(x^2+y^2\right)}y\right) \end{aligned}$$

$$f_{xx} = -2e^{-\left(x^2+y^2\right)}\left(1-2x^2\right)$$

$$f_{xy} = 4e^{-\left(x^2+y^2\right)}xy$$

$$f_{yx} = 4e^{-\left(x^2+y^2\right)}xy$$

$$f_{yy} = -2e^{-\left(x^2+y^2\right)}\left(1-2y^2\right)$$

$$H_f\left(x,y\right) = \begin{pmatrix} -2e^{-\left(x^2+y^2\right)}\left(1-2x^2\right) & 4e^{-\left(x^2+y^2\right)}xy \\ 4e^{-\left(x^2+y^2\right)}xy & -2e^{-\left(x^2+y^2\right)}\left(1-2y^2\right) \end{pmatrix}$$

(4)

$$\begin{aligned} \nabla f\left(x,y\right) &=^t \left(\partial_x f\left(x,y\right), \partial_y f\left(x,y\right)\right) \\ &=^t \left(\frac{x}{\sqrt{x^2+y^2+1}}, \frac{y}{\sqrt{x^2+y^2+1}}\right) \end{aligned}$$

$$f_{xx} = \frac{1}{\sqrt{x^2+y^2+1}} - \frac{x^2}{\left(x^2+y^2+1\right)^{\frac{3}{2}}}$$

$$f_{xy} = -\frac{xy}{\left(x^2+y^2+1\right)^{\frac{3}{2}}}$$

$$f_{yx} = -\frac{xy}{\left(x^2+y^2+1\right)^{\frac{3}{2}}}$$

$$f_{yy} = \frac{1}{\sqrt{x^2+y^2+1}} - \frac{y^2}{\left(x^2+y^2+1\right)^{\frac{3}{2}}}$$

$$H_f\left(x,y\right) = \begin{pmatrix} \frac{1}{\sqrt{x^2+y^2+1}} - \frac{x^2}{\left(x^2+y^2+1\right)^{\frac{3}{2}}} & -\frac{xy}{\left(x^2+y^2+1\right)^{\frac{3}{2}}} \\ -\frac{xy}{\left(x^2+y^2+1\right)^{\frac{3}{2}}} & \frac{1}{\sqrt{x^2+y^2+1}} - \frac{y^2}{\left(x^2+y^2+1\right)^{\frac{3}{2}}} \end{pmatrix}$$

$$
\begin{aligned}
f(0,b) &= f(1,0) + \sum_{k=1}^{3} \frac{1}{k!} \left(\left(\left({}^t(0,b) - {}^t(1,0) \right) \cdot \nabla \right)^k f \right) \left({}^t(1,0) \right) \\
&= f(1,0) + \sum_{k=1}^{3} \frac{1}{k!} \left(\left(\left({}^t(-1,b) \right) \cdot \nabla \right)^k f \right) \left({}^t(1,0) \right) \\
&= f(1,0) + \sum_{k=1}^{3} \frac{1}{k!} \left(\left(-\partial_x + b\partial_y \right)^k f \right) \left({}^t(1,0) \right) \\
&= \dots
\end{aligned}
$$

(5)

$$
\begin{aligned}
\nabla f(x,y) &=^t \left(\partial_x f(x,y), \partial_y f(x,y) \right) \\
&=^t \left(\frac{\sqrt{x^2+y^2}\left(\frac{2x\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^3\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} \right)}{x^2\cos\pi y}, \frac{\sqrt{x^2+y^2}\left(-\frac{\pi x^2\sin(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^2 y\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} \right)}{x^2\cos\pi y} \right)
\end{aligned}
$$

$$
\begin{aligned}
f_{xx} = &\frac{\sqrt{x^2+y^2}\left(-\frac{5x^2\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} + \frac{2\cos(\pi y)}{\sqrt{x^2+y^2}} + \frac{3x^4\cos(\pi y)}{(x^2+y^2)^{\frac{5}{2}}} \right)}{x^2\cos(\pi y)} \\
&- \frac{2\sqrt{x^2+y^2}\left(\frac{2x\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^3\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} \right)}{x^3\cos(\pi y)} \\
&+ \frac{\left(\frac{2x\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^3\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} \right)}{x\sqrt{x^2+y^2}\cos(\pi y)}
\end{aligned}
$$

$$f_{xy} = \frac{y\left(\frac{2x\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^3\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x^2\cos(\pi y)\sqrt{x^2+y^2}}$$

$$+ \frac{\sqrt{x^2+y^2}\left(-\frac{2\pi x\sin(\pi y)}{\sqrt{x^2+y^2}} - \frac{2xy\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} + \frac{\pi x^3\sin(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} + \frac{3x^3y\cos(\pi y)}{(x^2+y^2)^{\frac{5}{2}}}\right)}{x^2\cos(\pi y)}$$

$$+ \frac{\pi\sqrt{x^2+y^2}\sin(\pi y)\left(\frac{2x\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^3\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x^2\left(\cos(\pi y)\right)^2}$$

$$f_{yx} = \frac{y\left(\frac{2x\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^3\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x^2\cos(\pi y)\sqrt{x^2+y^2}}$$

$$+ \frac{\sqrt{x^2+y^2}\left(-\frac{2\pi x\sin(\pi y)}{\sqrt{x^2+y^2}} - \frac{2xy\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} + \frac{\pi x^3\sin(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} + \frac{3x^3y\cos(\pi y)}{(x^2+y^2)^{\frac{5}{2}}}\right)}{x^2\cos(\pi y)}$$

$$+ \frac{\pi\sqrt{x^2+y^2}\sin(\pi y)\left(\frac{2x\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^3\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x^2\left(\cos(\pi y)\right)^2}$$

$$f_{yy} = \frac{\sqrt{x^2+y^2}\left(\frac{2\pi x^2y\sin(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} - \frac{\pi^2x^2\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^2\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} + \frac{3x^2y^2\cos(\pi y)}{(x^2+y^2)^{\frac{5}{2}}}\right)}{x^2\cos(\pi y)}$$

$$- \frac{y\left(\frac{\pi x^2\sin(\pi y)}{\sqrt{x^2+y^2}} + \frac{x^2y\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x^2\sqrt{x^2+y^2}\cos(\pi y)}$$

$$- \frac{\pi\sqrt{x^2+y^2}\sin(\pi y)\left(\frac{\pi x^2\sin(\pi y)}{\sqrt{x^2+y^2}} + \frac{x^2y\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x^2\left(\cos(\pi y)\right)^2}$$

$$H_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

ここで、

$$f_{xx} = \frac{\sqrt{x^2+y^2}\left(-\frac{5x^2\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} + \frac{2\cos(\pi y)}{\sqrt{x^2+y^2}} + \frac{3x^4\cos(\pi y)}{(x^2+y^2)^{\frac{5}{2}}}\right)}{x^2\cos(\pi y)} - \frac{2\sqrt{x^2+y^2}\left(\frac{2x\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^3\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x^3\cos(\pi y)} + \frac{\left(\frac{2x\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^3\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x\sqrt{x^2+y^2}\cos(\pi y)}$$

$$f_{xy} = \frac{y\left(\frac{2x\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^3\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x^2\cos(\pi y)\sqrt{x^2+y^2}} + \frac{\sqrt{x^2+y^2}\left(-\frac{2\pi x\sin(\pi y)}{\sqrt{x^2+y^2}} - \frac{2xy\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} + \frac{\pi x^3\sin(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} + \frac{3x^3y\cos(\pi y)}{(x^2+y^2)^{\frac{5}{2}}}\right)}{x^2\cos(\pi y)} + \frac{\pi\sqrt{x^2+y^2}\sin(\pi y)\left(\frac{2x\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^3\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x^2\left(\cos(\pi y)\right)^2}$$

$$f_{yx} = \frac{y\left(\frac{2x\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^3\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x^2\cos(\pi y)\sqrt{x^2+y^2}} + \frac{\sqrt{x^2+y^2}\left(-\frac{2\pi x\sin(\pi y)}{\sqrt{x^2+y^2}} - \frac{2xy\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} + \frac{\pi x^3\sin(\pi y)}{(x^2+y^2)^{\frac{3}{2}}} + \frac{3x^3y\cos(\pi y)}{(x^2+y^2)^{\frac{5}{2}}}\right)}{x^2\cos(\pi y)} + \frac{\pi\sqrt{x^2+y^2}\sin(\pi y)\left(\frac{2x\cos(\pi y)}{\sqrt{x^2+y^2}} - \frac{x^3\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x^2\left(\cos(\pi y)\right)^2}$$

$$f_{yy}=\frac{\sqrt{x^2+y^2}\left(\frac{2\pi x^2 y\sin(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}-\frac{\pi^2 x^2\cos(\pi y)}{\sqrt{x^2+y^2}}-\frac{x^2\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}+\frac{3x^2y^2\cos(\pi y)}{(x^2+y^2)^{\frac{5}{2}}}\right)}{x^2\cos(\pi y)}$$

$$-\frac{y\left(\frac{\pi x^2\sin(\pi y)}{\sqrt{x^2+y^2}}+\frac{x^2 y\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x^2\sqrt{x^2+y^2}\cos(\pi y)}$$

$$-\frac{\pi\sqrt{x^2+y^2}\sin(\pi y)\left(\frac{\pi x^2\sin(\pi y)}{\sqrt{x^2+y^2}}+\frac{x^2 y\cos(\pi y)}{(x^2+y^2)^{\frac{3}{2}}}\right)}{x^2\left(\cos(\pi y)\right)^2}$$

(6)

$$\begin{aligned}\nabla f(x,y) &=^t\left(\partial_x f(x,y),\partial_y f(x,y)\right)\\ &=^t\left(2xe^{x^2}\cos\left(\pi y^3\right),-3\pi e^{x^2}y^2\sin\left(\pi y^3\right)\right)\end{aligned}$$

$$f_{xx}=2e^{x^2}\cos\left(\pi y^3\right)\left(1+2x^2\right)$$

$$f_{xy}=-6\pi xy^2e^{x^2}\sin\left(\pi y^3\right)$$

$$f_{yx}=-6\pi xy^2e^{x^2}\sin\left(\pi y^3\right)$$

$$f_{yy}=-3\pi ye^{x^2}\left(3\pi y^3\cos\left(\pi y^3\right)+2\sin\left(\pi y^3\right)\right)$$

$$H_f(x,y)=\begin{pmatrix}2e^{x^2}\cos\left(\pi y^3\right)\left(1+2x^2\right) & -6\pi xy^2e^{x^2}\sin\left(\pi y^3\right)\\ -6\pi xy^2e^{x^2}\sin\left(\pi y^3\right) & -3\pi ye^{x^2}\left(3\pi y^3\cos\left(\pi y^3\right)+2\sin\left(\pi y^3\right)\right)\end{pmatrix}$$

$$\begin{aligned}f(0,b) &= f(1,0)+\sum_{k=1}^{3}\frac{1}{k!}\left(\left(\left({}^t(0,b)-{}^t(1,0)\right)\cdot\nabla\right)^k f\right)\left({}^t(1,0)\right)\\ &= f(1,0)+\sum_{k=1}^{3}\frac{1}{k!}\left(\left(\left({}^t(-1,b)\right)\cdot\nabla\right)^k f\right)\left({}^t(1,0)\right)\\ &= f(1,0)+\sum_{k=1}^{3}\frac{1}{k!}\left(\left(-\partial_x+b\partial_y\right)^k f\right)\left({}^t(1,0)\right)\\ &= \ldots\end{aligned}$$

(7)

$$\begin{aligned}\nabla f(x,y) &= {}^t\left(\partial_x f(x,y), \partial_y f(x,y)\right) \\ &= {}^t\left(\frac{x}{(x^2+y^2+1)\sqrt{x^2+y^2}}, \frac{y}{(x^2+y^2+1)\sqrt{x^2+y^2}}\right)\end{aligned}$$

$$f_{xx} = \frac{1}{(x^2+y^2+1)\sqrt{x^2+y^2}}\left(1 - \frac{x^2}{x^2+y^2} - \frac{2x^2}{x^2+y^2+1}\right)$$

$$f_{xy} = -\frac{xy}{(x^2+y^2+1)\sqrt{x^2+y^2}}\left(\frac{1}{x^2+y^2} + \frac{2}{x^2+y^2+1}\right)$$

$$f_{yx} = -\frac{xy}{(x^2+y^2+1)\sqrt{x^2+y^2}}\left(\frac{1}{x^2+y^2} + \frac{2}{x^2+y^2+1}\right)$$

$$f_{yy} = \frac{1}{(x^2+y^2+1)\sqrt{x^2+y^2}}\left(1 - \frac{y^2}{x^2+y^2} - \frac{2y^2}{x^2+y^2+1}\right)$$

$$\begin{aligned}&H_f(x,y) \\ &= \frac{1}{(x^2+y^2+1)\sqrt{x^2+y^2}}\begin{pmatrix} 1 - \frac{x^2}{x^2+y^2} - \frac{2x^2}{x^2+y^2+1} & -xy\left(\frac{1}{x^2+y^2} + \frac{2}{x^2+y^2+1}\right) \\ -xy\left(\frac{1}{x^2+y^2} + \frac{2}{x^2+y^2+1}\right) & 1 - \frac{y^2}{x^2+y^2} - \frac{2y^2}{x^2+y^2+1} \end{pmatrix}\end{aligned}$$

(8)

$$\begin{aligned}\nabla f(x,y) &= {}^t\left(\partial_x f(x,y), \partial_y f(x,y)\right) \\ &= {}^t\left(2\pi x\cos^2(\pi y)\cos\left(\pi x^2\cos^2(\pi y)\right), -2\pi^2 x^2 \sin(\pi y)\cos(\pi y)\cos\left(\pi x^2\cos^2(\pi y)\right)\right)\end{aligned}$$

$$f_{xx} = 2\pi\cos^2(\pi y)\left(\cos\left(\pi x^2\cos^2(\pi y)\right) - 2\pi x^2\cos^2(\pi y)\sin\left(\pi x^2\cos^2(\pi y)\right)\right)$$

$$f_{xy} = 4\pi^2 x\sin(\pi y)\cos(\pi y)\left(\pi x^2\cos^2(\pi y)\sin\left(\pi x^2\cos^2(\pi y)\right) - \cos\left(\pi x^2\cos^2(\pi y)\right)\right)$$

$$f_{yx} = 4\pi^2 x \sin(\pi y)\cos(\pi y)\left(\pi x^2\cos^2(\pi y)\sin\left(\pi x^2\cos^2(\pi y)\right) - \cos\left(\pi x^2\cos^2(\pi y)\right)\right)$$

$$\begin{aligned}f_{yy} = &-2\pi^3 x^2\left(\cos^2(\pi y)\left(\cos\left(\pi x^2\cos^2(\pi y)\right) + 2\pi x^2\sin^2(\pi y)\sin\left(\pi x^2\cos^2(\pi y)\right)\right)\right)\\ &+ 2\pi^3 x^2\left(\sin^2(\pi y)\cos\left(\pi x^2\cos^2(\pi y)\right)\right)\end{aligned}$$

$$H_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

ここで、

$$f_{xx} = 2\pi\cos^2(\pi y)\left(\cos\left(\pi x^2\cos^2(\pi y)\right) - 2\pi x^2\cos^2(\pi y)\sin\left(\pi x^2\cos^2(\pi y)\right)\right)$$

$$f_{xy} = 4\pi^2 x \sin(\pi y)\cos(\pi y)\left(\pi x^2\cos^2(\pi y)\sin\left(\pi x^2\cos^2(\pi y)\right) - \cos\left(\pi x^2\cos^2(\pi y)\right)\right)$$

$$f_{yx} = 4\pi^2 x \sin(\pi y)\cos(\pi y)\left(\pi x^2\cos^2(\pi y)\sin\left(\pi x^2\cos^2(\pi y)\right) - \cos\left(\pi x^2\cos^2(\pi y)\right)\right)$$

$$\begin{aligned}f_{yy} = &-2\pi^3 x^2\left(\cos^2(\pi y)\left(\cos\left(\pi x^2\cos^2(\pi y)\right) + 2\pi x^2\sin^2(\pi y)\sin\left(\pi x^2\cos^2(\pi y)\right)\right)\right)\\ &+ 2\pi^3 x^2\left(\sin^2(\pi y)\cos\left(\pi x^2\cos^2(\pi y)\right)\right)\end{aligned}$$

A3.2

(1)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f\left(X(t), Y(t)\right) &= \frac{\partial}{\partial X}\frac{\mathrm{d}X(t)}{\mathrm{d}t} + \frac{\partial}{\partial Y}\frac{\mathrm{d}Y(t)}{\mathrm{d}t}\\ &= 2Xe^{X^2}\cos\left(\pi Y^3\right)\cdot\frac{1}{\sqrt{2}} - 3\pi e^{X^2} Y^2\sin\left(\pi Y^3\right)\cdot\frac{1}{\sqrt{2}}\\ &= te^{\frac{t^2}{2}}\cos\left(\frac{\sqrt{2}\pi}{4}t^3\right) - \frac{3\sqrt{2}}{4}\pi t^2 e^{\frac{t^2}{2}}\sin\left(\frac{\sqrt{2}\pi}{4}t^3\right)\\ &= te^{\frac{t^2}{2}}\left(\cos\left(\frac{\sqrt{2}\pi}{4}t^3\right) - \frac{3\sqrt{2}}{4}\pi t\sin\left(\frac{\sqrt{2}\pi}{4}t^3\right)\right)\end{aligned}$$

$$\lim_{t\to 0} te^{\frac{t^2}{2}}\left(\cos\left(\frac{\sqrt{2}\pi}{4}t^3\right) - \frac{3\sqrt{2}}{4}\pi t\sin\left(\frac{\sqrt{2}\pi}{4}t^3\right)\right) = 0\cdot 1\cdot 1 = 0$$

(2)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f\left(X\left(t\right), Y\left(t\right)\right) &= \frac{\partial}{\partial X}\frac{\mathrm{d}X\left(t\right)}{\mathrm{d}t} + \frac{\partial}{\partial Y}\frac{\mathrm{d}Y\left(t\right)}{\mathrm{d}t} \\
&= 2Xe^{X^2}\cos\left(\pi Y^3\right)\cdot\frac{1}{2} - 3\pi e^{X^2}Y^2\sin\left(\pi Y^3\right)\cdot\frac{\sqrt{3}}{2} \\
&= Xe^{X^2}\cos\left(\pi Y^3\right) - \frac{3\sqrt{3}}{2}\pi e^{X^2}Y^2\sin\left(\pi Y^3\right) \\
&= \frac{1+t}{2}e^{\frac{(1+t)^2}{4}}\cos\left(\pi\left(\frac{3+\sqrt{3}t}{2}\right)^3\right) \\
&\quad - \frac{3\sqrt{3}}{2}\pi e^{\frac{(1+t)^2}{4}}\left(\frac{3+\sqrt{3}t}{2}\right)^2\sin\left(\pi\left(\frac{3+\sqrt{3}t}{2}\right)^3\right) \\
&= e^{\frac{(1+t)^2}{4}}\left(\frac{1+t}{2}\cos\left(\pi\left(\frac{3+\sqrt{3}t}{2}\right)^3\right) - \frac{3\sqrt{3}}{2}\pi\left(\frac{3+\sqrt{3}t}{2}\right)^2\sin\left(\pi\left(\frac{3+\sqrt{3}t}{2}\right)^3\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{t\to 0} e^{\frac{(1+t)^2}{4}}\left(\frac{1+t}{2}\cos\left(\pi\left(\frac{3+\sqrt{3}t}{2}\right)^3\right) - \frac{3\sqrt{3}}{2}\pi\left(\frac{3+\sqrt{3}t}{2}\right)^2\sin\left(\pi\left(\frac{3+\sqrt{3}t}{2}\right)^3\right)\right) \\
&= e^{\frac{1}{4}}\left(\frac{1}{2}\cos\left(\frac{27}{8}\pi\right) - \frac{27\sqrt{3}}{8}\pi\sin\left(\frac{27}{8}\pi\right)\right) \\
&= \dots
\end{aligned}
$$

(3)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f\left(X\left(t\right), Y\left(t\right)\right) &= \frac{\partial}{\partial X}\frac{\mathrm{d}X\left(t\right)}{\mathrm{d}t} + \frac{\partial}{\partial Y}\frac{\mathrm{d}Y\left(t\right)}{\mathrm{d}t} \\
&= 2Xe^{X^2}\cos\left(\pi Y^3\right)\cdot 2t - 3\pi e^{X^2}Y^2\sin\left(\pi Y^3\right)\cdot e^t \\
&= e^{\left(t^2\right)^2}\left(4t\cdot t^2\cdot\cos\left(\pi\left(e^t\right)^3\right) - 3\pi\cdot e^t\cdot\left(e^t\right)^2\cdot\sin\left(\pi\left(e^t\right)^3\right)\right) \\
&= e^{t^4}\left(4t^3\cos\left(e^{3t}\pi\right) - 3\pi e^{3t}\sin\left(e^{3t}\pi\right)\right)
\end{aligned}
$$

(4)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f\left(X\left(t\right), Y\left(t\right)\right) &= \frac{\partial}{\partial X}\frac{\mathrm{d}X\left(t\right)}{\mathrm{d}t} + \frac{\partial}{\partial Y}\frac{\mathrm{d}Y\left(t\right)}{\mathrm{d}t} \\ &= 2Xe^{X^2}\cos\left(\pi Y^3\right)\cdot\left(-\sin\left(t\right)\right) - 3\pi e^{X^2}Y^2\sin\left(\pi Y^3\right)\cdot\cos\left(t\right) \\ &= e^{\cos^2\left(t\right)}\left(-2\sin\left(t\right)\cos\left(t\right)\cos\left(\pi\sin^3\left(t\right)\right) - 3\pi\sin^2\left(t\right)\cos\left(t\right)\sin\left(\pi\sin^3\left(t\right)\right)\right) \\ &= -e^{\cos^2\left(t\right)}\sin\left(t\right)\cos\left(t\right)\left(2\cos\left(\pi\sin^3\left(t\right)\right) + 3\pi\sin\left(t\right)\sin\left(\pi\sin^3\left(t\right)\right)\right)\end{aligned}$$

B3.3

$\phi\left(x, y\right) = f\left(x, y\right) - f\left(x, b\right)$とおく、十分小さい$h, k$をとり、$\phi\left(x, b+k\right)$に平均値の定理を使うと

$$\begin{aligned}&\phi\left(a+h, b+k\right) - \phi\left(a, b+k\right) \\ &= h\phi_x\left(a+\theta h, b+k\right) \qquad \left(0 < \theta < 1\right) \\ &= h\left(f_x\left(a+\theta h, b+k\right) - f_x\left(a+\theta h, b\right)\right)\end{aligned}$$

また、$f_x\left(a+\theta h, y\right)$に使うと

$$\begin{aligned}\phi\left(a+h, b+k\right) - \phi\left(a, b+k\right) &= hkf_x\left(a+\theta h, b+\rho k\right) \\ &\quad where\ 0 < \rho < 1\end{aligned}$$

両辺をkで割って$k \to 0$での極限をとると

$$f_y\left(a+h, b\right) - f_y\left(a, b\right) = h\lim_{k\to 0} f_{xy}\left(a+\theta h, b+\rho k\right)$$

もう一度hで割って$h \to 0$での極限をとると

$$\begin{aligned}f_{yx}\left(a, b\right) &= \lim_{h\to 0}\frac{f_y\left(a+h, b\right) - f_y\left(a, b\right)}{h} \\ &= \lim_{\substack{h\to 0 \\ k\to 0}} f_{xy}\left(a+\theta h, b+\rho k\right) = f_{xy}\left(a, b\right)\end{aligned}$$

B3.4

$f \in \mathbb{R}^2 : continuous$
$\Longleftrightarrow \forall x_0, \forall \epsilon > 0, \exists \delta > 0, \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon$
$(x_0 :={}^t(a,b)$とする$)$
ここで、$\partial_x f(x,y)$と$\partial_x f(x,y)$の連続性より
$\lim_{h\to 0} \frac{f(a+h,b)}{h} = 0, \lim_{k\to 0} \frac{f(a,b+k)}{k} = 0$
$\lim_{\substack{h\to 0\\ k\to 0}} \frac{f(a+h,b+k) - f(a,b)}{hk} = 0$
$i.e. \forall \epsilon > 0, \exists \delta > 0, \|{}^t(x,y) - {}^t(a,b)\| < \delta \Rightarrow \|f\left({}^t(x,y)\right) - f\left({}^t(a,b)\right)\| < \epsilon$

B3.5

${}^t(x,y) \neq {}^t(0,0)$
$f_x(x,y) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$
$f_y(x,y) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0$
よって、軸方向では偏微分可能である
$y = x$方向で${}^t(0,0)$に近づくと
$\lim_{{}^t(x,y)\to {}^t(0,0)} f(x,y) = \lim_{x\to 0} \frac{2x^2}{x^2+x^2} = 1 \neq 0$
すなわち、fは${}^t(0,0)$で連続でない

B3.6

存在性は自明だから略にして、可換性がないことだけ証明する

$$\begin{aligned}\partial_y \partial_x f(0,0) &= \partial_y \frac{y\left(x^4 + 4x^2y^2 - y^4\right)}{\left(x^2+y^2\right)^2} \\ &= \lim_{k\to 0} \frac{\frac{-k^5}{k^4} - 0}{k} = -1\end{aligned}$$

$$\begin{aligned}\partial_x \partial_y f(0,0) &= \partial_x \frac{x\left(x^4 - 4x^2y^2 - y^4\right)}{\left(x^2+y^2\right)^2} \\ &= \lim_{h\to 0} \frac{\frac{x^5}{h^4} - 0}{h} = 1\end{aligned}$$

よって、偏微分の順番は可換ではない
元の式から考えると、可換できない理由はyの符号は負であるから、順番を変化すると符号も変化する、また、図をみるとこれは反時計回りの回転対称で、普通の対称ではない

B3.7

帰納法で考えよう
$m=1$のときは自明で略
$m=n$のとき成立すると仮定すると
$$f(b)=f(a)+\sum_{k=1}^{n-1}\frac{1}{k!}\left(((b-a)\cdot\nabla)^{k}f\right)(a)+\frac{1}{n!}\int_0^1\left(((b-a)\cdot\nabla)^{n}f\right)(a+\theta(b-a))(1-\theta)^{n-1}\,\mathrm{d}\theta$$
があって
ここで$c:=\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}b_1\\b_2\end{pmatrix}-\begin{pmatrix}a_1\\a_2\end{pmatrix}$とし、
$g(\theta)=f(a+\theta(b-a))=f(a+\theta c)$とする
$$\int_0^1\left(((b-a)\cdot\nabla)^{n}f\right)(a+\theta(b-a))(1-\theta)^{n-1}\,\mathrm{d}\theta$$
$$=\int_0^1\left((c\cdot\nabla)^{n}f\right)(a+\theta c)(1-\theta)^{n-1}\,\mathrm{d}\theta$$
$$=\int_0^1\left((c\cdot\nabla)^{n}g\right)\theta(1-\theta)^{n-1}\,\mathrm{d}\theta$$

$$g(1)=f(b)=g(0)+\sum_{k=1}^{n-1}\frac{1}{k!}\left((c\cdot\nabla)^{k}g\right)(0)+\frac{1}{n!}\int_0^1\left((c\cdot\nabla)^{n}g\right)\theta(1-\theta)^{n-1}\,\mathrm{d}\theta$$
$$=g(0)+\sum_{k=1}^{n-1}\frac{1}{k!}\left((c\cdot\nabla)^{k}g\right)(0)$$
$$+\frac{1}{n!}\left(\left[-\frac{\left((c\cdot\nabla)^{n}g\right)\theta(1-\theta)^{n}}{n}\right]_0^1+\frac{1}{n+1}\int_0^1\left((c\cdot\nabla)^{n+1}g\right)\theta(1-\theta)^{n}\,\mathrm{d}\theta\right)$$
$$=g(0)+\sum_{k=1}^{n-1}\frac{1}{k!}\left((c\cdot\nabla)^{k}g\right)(0)+\frac{1}{n!}\left((c\cdot\nabla)^{n}g\right)(0)$$
$$+\frac{1}{(n+1)!}\int_0^1\left((c\cdot\nabla)^{n+1}g\right)(\theta)(1-\theta)^{n}\,\mathrm{d}\theta$$
$$=g(0)+\sum_{k=1}^{n}\frac{1}{k!}\left((c\cdot\nabla)^{k}g\right)(0)+\frac{1}{(n+1)!}\int_0^1\left((c\cdot\nabla)^{n+1}g\right)(\theta)(1-\theta)^{n}\,\mathrm{d}\theta$$

これは$m=n+1$のときだから、帰納法より成り立つ

B3.8

(1)

$$
\begin{aligned}
&\partial_r \partial_s v(r,s) = 0 \\
&\overset{\int \partial \mathrm{r}}{\Longleftrightarrow} \partial_s v(r,s) = g'(s) \\
&\overset{\int \partial \mathrm{s}}{\Longleftrightarrow} v(r,s) = f(r) + g(s)
\end{aligned}
$$

実際、ここでの$g'(s)$はただの記号で、sと関係するrと関係しない関数だけ表している、普通の$g(s)$でかけてもいいだが、形式の美観のためここで微分の形式で書く

(2)

$$
\begin{aligned}
\partial_t^2 v(r(t,x), s(t,x)) &= \partial_t (\partial_t v(r,s)) \\
&= \partial_t \left(\frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial t} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial t} \right) \\
&= \partial_t \left(\frac{\partial v}{\partial r} \cdot (1) + \frac{\partial v}{\partial s} \cdot (-1) \right) \\
&= \partial_t \left(\frac{\partial v}{\partial r} - \frac{\partial v}{\partial s} \right) \\
&= \left(\frac{\partial^2 v}{\partial r^2} \cdot \frac{\partial r}{\partial t} \right) - \left(\frac{\partial^2 v}{\partial s^2} \cdot \frac{\partial s}{\partial t} \right) \\
&= \left(\frac{\partial^2 v}{\partial r^2} \cdot (1) \right) - \left(\frac{\partial^2 v}{\partial s^2} \cdot (-1) \right) \\
&= \frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 v}{\partial s^2}
\end{aligned}
$$

$$
\begin{aligned}
\partial_x^2 v\left(r\left(x,t\right),s\left(x,t\right)\right) &= \partial_x\left(\frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial x}\right)\\
&= \partial_x\left(\frac{\partial v}{\partial r}\cdot(1)+\frac{\partial v}{\partial s}\cdot(1)\right)\\
&= \partial_x\left(\frac{\partial v}{\partial r}+\frac{\partial v}{\partial s}\right)\\
&= \left(\frac{\partial^2 v}{\partial r^2}\cdot\frac{\partial r}{\partial x}\right)+\left(\frac{\partial^2 v}{\partial s^2}\cdot\frac{\partial s}{\partial x}\right)\\
&= \frac{\partial^2 v}{\partial r^2}+\frac{\partial^2 v}{\partial s^2}
\end{aligned}
$$

(3)

$$
\begin{aligned}
0 &= \partial_t^2 u\left(t,x\right)-\partial_x^2 u\left(t,x\right)\\
&= \left(\partial_t u\left(t,x\right)+\partial_x u\left(t,x\right)\right)\left(\partial_t u\left(t,x\right)-\partial_x u\left(t,x\right)\right)
\end{aligned}
$$

ここで置換を考えよう
$a : x+t, b := x-t$とすると、$x=\frac{1}{2}\left(a+b\right), t=\frac{1}{2}\left(a-b\right)$で
$\frac{\partial u}{\partial a}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial a}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial a}=\frac{1}{2}\left(\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\right)$
$\frac{\partial u}{\partial b}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial b}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial b}=-\frac{1}{2}\left(\frac{\partial u}{\partial t}-\frac{\partial u}{\partial x}\right)$
言い換えれば、$\frac{\partial u}{\partial a}\cdot\frac{\partial u}{\partial b}=0$、これは(1)より確実に成立する

B3.9

(1)

$$
\begin{aligned}
f''\left(r\right)+\frac{f'\left(r\right)}{r} &= 0\\
\frac{f''\left(r\right)}{f'\left(r\right)} &= -\frac{1}{r}\\
\overset{\int \mathrm{d}r}{\iff} \log\left|f'\left(r\right)\right| &= -\log\left|r\right|+C_1\\
\iff f'\left(r\right) &= \frac{e^{C_1}}{r}\\
\overset{\int \mathrm{d}r}{\iff} f\left(r\right) &= e^{C_1}\log\left|r\right|+C_2
\end{aligned}
$$

(2)

$$
\begin{aligned}
\nabla f\left(r\left(x,y\right)\right) &= {}^t\left(\partial_x f\left(r\left(x,y\right)\right), \partial_y f\left(r\left(x,y\right)\right)\right) \\
&= {}^t\left(\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x}, \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y}\right) \\
&= \begin{pmatrix} \dfrac{xf'\left(r\right)}{\sqrt{x^2+y^2}} \\ \dfrac{yf'\left(r\right)}{\sqrt{x^2+y^2}} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{f'\left(r\right)x}{r} \\ \dfrac{f'\left(r\right)y}{r} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{f'\left(r\right)x}{r}\right) &= \frac{\partial}{\partial x}f'\left(r\right)\cdot\frac{x}{r} + f'\left(r\right)\cdot\frac{\partial}{\partial x}\left(\frac{x}{r}\right) \\
&= f''\left(r\right)\cdot\frac{x}{r}\cdot\frac{x}{r} + f'\left(r\right)\cdot\frac{y^2}{r^3} \\
&= \frac{x^2}{r^2}f''\left(r\right) + \frac{y^2}{r^3}f'\left(r\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{f'\left(r\right)x}{r}\right) &= \frac{\partial}{\partial y}f'\left(r\right)\cdot\frac{x}{r} + f'\left(r\right)\cdot\frac{\partial}{\partial y}\left(\frac{x}{r}\right) \\
&= f''\left(r\right)\cdot\frac{xy}{r^2} - f'\left(r\right)\cdot\frac{xy}{r^3} \\
&= \frac{xy}{r^2}f''\left(r\right) - \frac{xy}{r^3}f'\left(r\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{f'\left(r\right)y}{r}\right) &= \frac{\partial}{\partial y}f'\left(r\right)\cdot\frac{y}{r} + f'\left(r\right)\cdot\frac{\partial}{\partial y}\left(\frac{y}{r}\right) \\
&= \frac{y^2}{r^2}f''\left(r\right) + \frac{x^2}{r^3}f'\left(r\right)
\end{aligned}
$$

$$
H_f\left(x,y\right) = \begin{pmatrix} \dfrac{x^2}{r^2}f''\left(r\right) + \dfrac{y^2}{r^3}f'\left(r\right) & \dfrac{xy}{r^2}f''\left(r\right) - \dfrac{xy}{r^3}f'\left(r\right) \\ \dfrac{xy}{r^2}f''\left(r\right) - \dfrac{xy}{r^3}f'\left(r\right) & \dfrac{y^2}{r^2}f''\left(r\right) + \dfrac{x^2}{r^3}f'\left(r\right) \end{pmatrix}
$$

(3)

(2)より、$\partial_x^2 f\left(x,y\right)=\dfrac{x^2}{r^2}f''\left(r\right)+\dfrac{y^2}{r^3}f'\left(r\right), \partial_y^2 f\left(x,y\right)=\dfrac{y^2}{r^2}f''\left(r\right)+\dfrac{x^2}{r^3}f'\left(r\right)$
よって

$$
\begin{aligned}
&\frac{x^2}{r^2}f''\left(r\right)+\frac{y^2}{r^3}f'\left(r\right)+\frac{y^2}{r^2}f''\left(r\right)+\frac{x^2}{r^3}f'\left(r\right)=0\\
&f''\left(r\right)+\frac{f'\left(r\right)}{r}=0
\end{aligned}
$$

これは(1)より、$f=c_1\log r+c_2=c_1\log\sqrt{x^2+y^2}+c_2$であることが成立する