P7.1

$\Delta := \left[\frac{k-1}{n}, \frac{k}{n}\right]_{1 \leq k \leq n}$ とする

$$
\begin{aligned}
\underline{\int_0^1} f(x)\,\mathrm{d}x &= \sup\{s(f, \Delta)\} \\
&= \sup\left\{\sum_{k=1}^{n} \inf_{x_k \in I_k}\{f(x_k)\}|I_k|\right\} \\
&= \sup\left\{\sum_{k=1}^{n} 0 \cdot \frac{1}{n}\right\} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\overline{\int_0^1} f(x)\,\mathrm{d}x &= \inf\{S(f, \Delta)\} \\
&= \inf\left\{\sum_{k=1}^{n} \sup_{x_k \in I_k}\{f(x_k)\}|I_k|\right\} \\
&= \inf\left\{\sum_{k=1}^{n} 1 \cdot \frac{1}{n}\right\} \\
&= 1
\end{aligned}
$$

$\underline{\int_0^1} f(x)\,\mathrm{d}x \neq \overline{\int_0^1} f(x)\,\mathrm{d}x$から、$f$が積分できない

A7.1

(1)

$$
\begin{aligned}
\iint_{[-1,1]\times[0,1]} x^2 e^y \mathrm{d}x\mathrm{d}y &= \int_0^1 e^y \left(\int_{-1}^1 x^2 \mathrm{d}x\right) \mathrm{d}y \\
&= \frac{2}{3}\int_0^1 e^y \mathrm{d}y \\
&= \frac{2}{3}(e-1)
\end{aligned}
$$

(2)

$$\iint_{[0,2]\times[0,1]} xy\cos y^2 \mathrm{d}x\mathrm{d}y = \int_0^1 y\cos y^2 \left(\int_0^2 x\mathrm{d}x\right)\mathrm{d}y$$
$$= 2\int_0^1 y\cos y^2 \mathrm{d}y$$
$$= \left[\sin y^2\right]_0^1$$
$$= \sin 1$$

(3)

$$\iint_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{3}]} (x\sin y - y\sin x)\,\mathrm{d}x\mathrm{d}y$$
$$= \iint_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{3}]} x\sin y\mathrm{d}x\mathrm{d}y - \iint_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{3}]} y\sin x\mathrm{d}x\mathrm{d}y$$
$$= \int_0^{\frac{\pi}{3}} \sin y \left(\int_0^{\frac{\pi}{2}} x\mathrm{d}x\right)\mathrm{d}y - \int_0^{\frac{\pi}{2}} \sin x \left(\int_0^{\frac{\pi}{3}} y\mathrm{d}y\right)\mathrm{d}x$$
$$= \frac{\pi^2}{8}\int_0^{\frac{\pi}{3}} \sin y\mathrm{d}y - \frac{\pi^2}{18}\int_0^{\frac{\pi}{2}} \sin x\mathrm{d}x$$
$$= -\frac{\pi^2}{8}\left[\cos y\right]_0^{\frac{\pi}{3}} + \frac{\pi^2}{18}\left[\cos x\right]_0^{\frac{\pi}{2}}$$
$$= -\frac{\pi^2}{8}\left(\frac{1}{2} - 1\right) + \frac{\pi^2}{18}(0-1)$$
$$= \frac{\pi^2}{16} - \frac{\pi^2}{18} = \frac{\pi^2}{144}$$

A7.2

(1)

$$\begin{aligned}\int_0^2 \int_0^1 xy^2 \mathrm{d}x\mathrm{d}y &= \int_0^2 y^2 \left(\int_0^1 x\mathrm{d}x \right) \mathrm{d}y \\ &= \frac{1}{2} \int_0^2 y^2 \mathrm{d}y \\ &= \frac{1}{6} (8-0) \\ &= \frac{4}{3}\end{aligned}$$

(2)

$$\begin{aligned}\int_0^1 \int_0^2 xy^2 \mathrm{d}y\mathrm{d}x &= \int_0^1 x \left(\int_0^2 y^2\mathrm{d}y \right) \mathrm{d}x \\ &= \frac{8}{3} \int_0^1 x \mathrm{d}x \\ &= \frac{4}{3}\end{aligned}$$

(3)

$$\begin{aligned}\int_{-1}^2 \int_0^1 2x \mathrm{d}x\mathrm{d}y &= \int_{-1}^2 \left(2 \int_0^1 x\mathrm{d}x \right) \mathrm{d}y \\ &= \int_{-1}^2 1 \mathrm{d}y \\ &= 3\end{aligned}$$

(4)

$$\begin{aligned}\int_0^2 \int_{-1}^1 e^{x+y}\mathrm{d}x\mathrm{d}y &= \int_0^2 e^y \left(\int_{-1}^1 e^x \mathrm{d}x\right) \mathrm{d}y \\ &= \left(e - \frac{1}{e}\right) \int_0^2 e^y \mathrm{d}y \\ &= \left(e - \frac{1}{e}\right) \left(e^2 - 1\right) \\ &= e^3 - e - e + \frac{1}{e} = e^3 - 2e + \frac{1}{e}\end{aligned}$$

(5)

$$\begin{aligned}\int_1^2 \int_0^2 2y \log x \mathrm{d}y\mathrm{d}x &= \int_1^2 \log x \left(\int_0^2 2y\mathrm{d}y\right) \mathrm{d}x \\ &= 4 \int_1^2 \log x \mathrm{d}x \\ &= 4 \left[x \left(\log x - 1\right)\right]_1^2 \\ &= 4 \left(2 \left(\log 2 - 1\right)\right) = 8 \left(\log 2 - 1\right)\end{aligned}$$

(6)

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos\left(x + y\right)\mathrm{d}x\mathrm{d}y &= \int_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{2}} \cos\left(x + y\right)\mathrm{d}x\right) \mathrm{d}y \\ &= \int_0^{\frac{\pi}{2}} \left[\sin\left(x + y\right)\right]_0^{\frac{\pi}{2}} \mathrm{d}y \\ &= \int_0^{\frac{\pi}{2}} \left(-\cos y - \sin y\right) \mathrm{d}y \\ &= -\left[\sin x - \cos x\right]_0^{\frac{\pi}{2}} \\ &= -\left(1 - \left(0 - 1\right)\right) \\ &= -2\end{aligned}$$

(7)

$$\begin{aligned}
\int_1^2 \int_1^2 \frac{1}{x^2y+y^3}\mathrm{d}x\mathrm{d}y &= \int_1^2 \left(\int_1^2 \frac{1}{x^2y+y^3}\right)\mathrm{d}y \\
&= \int_1^2 \left(\frac{\arctan\frac{2}{y} - \arctan\frac{1}{y}}{y^2}\right)\mathrm{d}y \\
&= \int_1^2 \frac{\arctan\frac{2}{y}}{y^2}\mathrm{d}y - \int_1^2 \frac{\arctan\frac{1}{y}}{y^2}\mathrm{d}y \\
&= -\frac{1}{2}\int_2^1 \arctan s\mathrm{d}s + \int_1^{\frac{1}{2}} \arctan t\mathrm{d}t \\
&= -\frac{1}{8}\left(\pi - 8\arctan 2 + \log\frac{25}{4}\right) + \frac{1}{4}\left(-\pi + 2\arctan\frac{1}{2} + \log\frac{64}{25}\right) \\
&= \frac{1}{32}\left(-3\pi + 4\arctan\frac{1}{2} + 8\arctan 2 + \log\frac{16384}{15625}\right)
\end{aligned}$$

(8)

$$\begin{aligned}
\int_2^4 \int_1^2 \frac{1}{(x+y)^2}\mathrm{d}x\mathrm{d}y &= \int_2^4 \left(\int_1^2 \frac{1}{(x+y)^2}\mathrm{d}x\right)\mathrm{d}y \\
&= \int_2^4 \left(\frac{1}{y+1} - \frac{1}{y+2}\right)\mathrm{d}y \\
&= \left[\log\frac{y+1}{y+2}\right]_2^4 \\
&= \log 10 - 2\log 3
\end{aligned}$$

A7.3

$$
\begin{aligned}
\text{LHS} &= \int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} \mathrm{d}x \right) \mathrm{d}y \\
&= \int_0^1 \left(\int_0^1 \frac{x^2}{(x^2+y^2)^2} \mathrm{d}x - \int_0^1 \frac{y^2}{(x^2+y^2)^2} \mathrm{d}x \right) \mathrm{d}y \\
&= \lim_{\substack{a\to 0\\ b\to 0}} \int_b^1 \left(\int_a^1 \frac{x^2}{(x^2+y^2)^2} \mathrm{d}x - \int_a^1 \frac{y^2}{(x^2+y^2)^2} \mathrm{d}x \right) \mathrm{d}y \\
&= \lim_{b\to 0} \int_b^1 \left(\lim_{a\to 0} \left(\int_a^1 \frac{x^2}{(x^2+y^2)^2} - \int_a^1 \frac{y^2}{(x^2+y^2)^2} \right) \right) \mathrm{d}y \\
&= \lim_{b\to 0} \int_b^1 \left(\lim_{a\to 0} \left[-\frac{x}{x^2+y^2} \right]_a^1 \right) \mathrm{d}y \\
&= \lim_{b\to 0} \int_b^1 \left(-\frac{1}{y^2+1} + \lim_{a\to 0} \frac{a}{y^2+a^2} \right) \mathrm{d}y \\
&= -\lim_{b\to 0} \int_b^1 \frac{1}{y^2+1} \mathrm{d}y + \lim_{\substack{a\to 0\\ b\to 0}} \int_b^1 \frac{a}{y^2+a^2} \mathrm{d}y \\
&= -\lim_{b\to 0} [\arctan y]_b^1 + \lim_{\substack{a\to 0\\ b\to 0}} \left[\arctan \frac{y}{a} \right]_b^1 \\
&= \arctan 1 + \lim_{\substack{a\to 0\\ b\to 0}} \left(\arctan \frac{1}{a} - \arctan \frac{b}{a} \right) \\
&= \arctan 1 + \lim_{\substack{a\to 0\\ b\to 0}} \arctan \frac{1}{a} - \lim_{\substack{a\to 0\\ b\to 0}} \arctan \frac{b}{a} \\
&= \arctan 1 + \frac{\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{RHS} &= \int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} \mathrm{d}y \right) \mathrm{d}x \\
&= \lim_{a \to 0} \int_a^1 \left(\lim_{b \to 0} \left[\frac{y}{x^2 + y^2} \right]_b^1 \right) \mathrm{d}x \\
&= \lim_{a \to 0} \int_a^1 \left(\frac{1}{x^2 + 1} - \lim_{b \to 0} \frac{b}{x^2 + b^2} \right) \mathrm{d}x \\
&= \lim_{a \to 0} \int_a^1 \frac{1}{x^2 + 1} \mathrm{d}x - \lim_{\substack{a \to 0 \\ b \to 0}} \int_a^1 \frac{b}{x^2 + b^2} \mathrm{d}x \\
&= \lim_{a \to 0} \left[\arctan x \right]_a^1 - \lim_{\substack{a \to 0 \\ b \to 0}} \left[\arctan \frac{x}{b} \right]_a^1 \\
&= \arctan 1 - \lim_{\substack{a \to 0 \\ b \to 0}} \arctan \frac{1}{b} + \lim_{\substack{a \to 0 \\ b \to 0}} \arctan \frac{a}{b} \\
&= \arctan 1 - \frac{\pi}{2}
\end{aligned}
$$

$\Longrightarrow \mathrm{LHS} \neq \mathrm{RHS}$

B7.4

fはIで可積であるから$\underline{\int}_I f = \overline{\int}_I f$で、$\Delta_{I'} \subset \Delta_I$から

$\underline{\int}_{I'} f = \sup \left\{ s\left(f, \Delta_{I'}\right) \right\} < \sup \left\{ s\left(f, \Delta_I\right) \right\} = \underline{\int}_I f$

$\overline{\int}_{I'} f = \inf \left\{ S\left(f, \Delta_{I'}\right) \right\} > \inf \left\{ S\left(f, \Delta_I\right) \right\} = \overline{\int}_I f$

さらに$\underline{\int}_{I'} f \leq \overline{\int}_{I'} f$から

$\underline{\int}_{I'} f = \overline{\int}_{I'} f$

すなわち、I'上でも可積である

B7.5

$$
\begin{aligned}
h : I \times J &\longrightarrow \mathbb{R} \\
(x, y) &\mapsto f(x)\, g(y)
\end{aligned}
$$

hは過程より連続

$$\begin{aligned}\int_{I\times J} f(x)\,g(y)\,\mathrm{d}x\mathrm{d}y &= \int_{I\times J} h(x,y)\,\mathrm{d}x\mathrm{d}y \\ &= \int_J \left(\int_I h(x,y)\,\mathrm{d}x\right)\mathrm{d}y \\ &= \int_J g(y)\left(\int_I f(x)\,\mathrm{d}x\right)\mathrm{d}y \\ &= \int_I f(x)\,\mathrm{d}x \int_J g(y)\,\mathrm{d}y\end{aligned}$$

B7.6

$$\int_a^b \int_a^b F\mathrm{d}x\mathrm{d}y$$
$$= \int_a^b \int_a^b f^2(x)\,g^2(y)\,\mathrm{d}x\mathrm{d}y - 2\int_a^b \int_a^b f(x)\,g(y)\,f(y)\,g(x)\,\mathrm{d}x\mathrm{d}y + \int_a^b \int_a^b f^2(y)\,g^2(x)\,\mathrm{d}x\mathrm{d}y$$
$$= 2\int_a^b f^2(x)\,\mathrm{d}x \int_a^b g^2(x)\,\mathrm{d}x - 2\left(\int_a^b f(x)\,g(x)\,\mathrm{d}x\right)^2 \geq 0$$

よって

$$\left(\int_a^b f(x)\,g(x)\,\mathrm{d}x\right)^2 \leq \int_a^b f^2(x)\,\mathrm{d}x \int_a^b g^2(x)\,\mathrm{d}x$$

B7.7

(1)

$$S(f,\Delta_1) = \left(\frac{3}{2}+\frac{7}{2}+2+4\right)\cdot\frac{1}{2} = \frac{11}{2}$$
$$s(f,\Delta_1) = \left(-1+1-\frac{1}{2}+\frac{3}{2}\right)\cdot\frac{1}{2} = \frac{1}{2}$$

(2)

$$S(f,\Delta) = \sup_{I_k\in\Delta} \sum_{x_k,y_k\in I_k} (x_k+2y_k+1)\,|I_k|$$

$$s\left(f,\Delta\right)=\inf_{I_k\in\Delta}\sum_{x_k,y_k\in I_k}\left(x_k+2y_k+1\right)\left|I_k\right|$$

(3)

$$\begin{aligned}\iint_{I\times J}\left(x+2y-1\right)\mathrm{d}x\mathrm{d}y&=\underline{\iint}_{I\times J}\left(x+2y-1\right)\mathrm{d}x\mathrm{d}y\\&=\sup s\left(x+2y-1,\Delta\right)\\&=\sup\sum_{\substack{i=1\\j=1}}^{\substack{i=n\\j=m}}\inf_{\substack{x_i\in I\\y_j\in J}}\left(x_i+2y_j-1\right)\left|\frac{\left|I\right|\left|J\right|}{mn}\right|\\&=\sup\sum_{\substack{i=1\\j=1}}^{\substack{i=n\\j=m}}\inf\left(\frac{\left(i-1\right)I}{n}+2\frac{\left(j-1\right)J}{m}-1\right)\left|\frac{\left|I\right|\left|J\right|}{mn}\right|\end{aligned}$$

B7.8

$0<y<\dfrac{1}{2}$の場合では、$f\left(q\right)<f\left(p\right),p\in\mathbb{Q},q\notin\mathbb{Q}$であるから

$D:=\left[0,\dfrac{1}{2}\right]\times\left[0,\dfrac{1}{2}\right]$の部分を考えよう

稠密性より、任意の分割Δに対して

$$s\left(f,\Delta\right)=\sum_{j=1}^{j=m}\inf f\left(x,y\right)\left|D_j\right|=\sum_{j=1}^{j=m}2y_j\left|D_j\right|$$

$$<\sum_{j=1}^{j=m}1\left|D_j\right|=\sum_{j=1}^{j=m}\sup f\left(x,y\right)\left|D_j\right|=S\left(f,\Delta\right)$$

すなわち、$\overline{\int}_D f=\inf S\left(f,\Delta\right)>\sup s\left(f,\Delta\right)=\underline{\int}_D f$から、$\overline{\int}_D f\neq\underline{\int}_D f$

積分できない

累次積分について、$x\in\mathbb{Q}$や$x\notin\mathbb{Q}$の場合は

$$\begin{cases}\int_0^1 f\mathrm{d}y\mathrm{d}x=\int_0^1 1\mathrm{d}x=1 & x\in\mathbb{Q}\\ \int_0^1 f\mathrm{d}y\mathrm{d}x=\int_0^1\int_0^1 2y\mathrm{d}y\mathrm{d}x=\int_0^1\mathrm{d}x=1 & x\notin\mathbb{Q}\end{cases}$$ よって、任意のx,yに

ついて、この累次積分は1である

B7.9

リベーグ