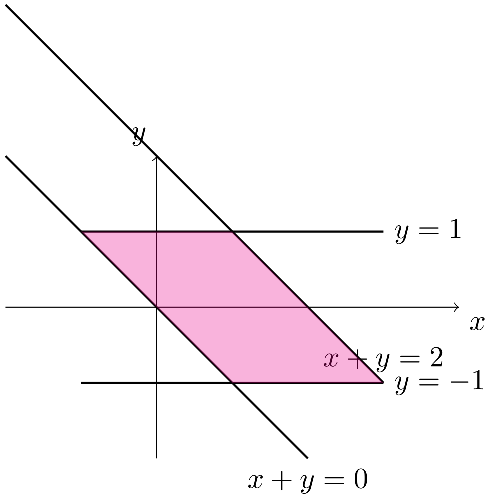
第 9 回

🐈‍⬛

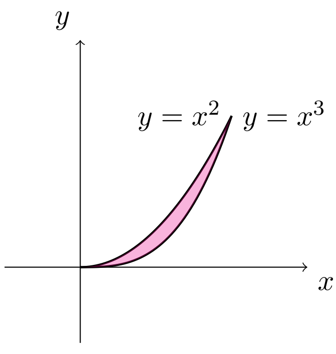
A8.1

(1)



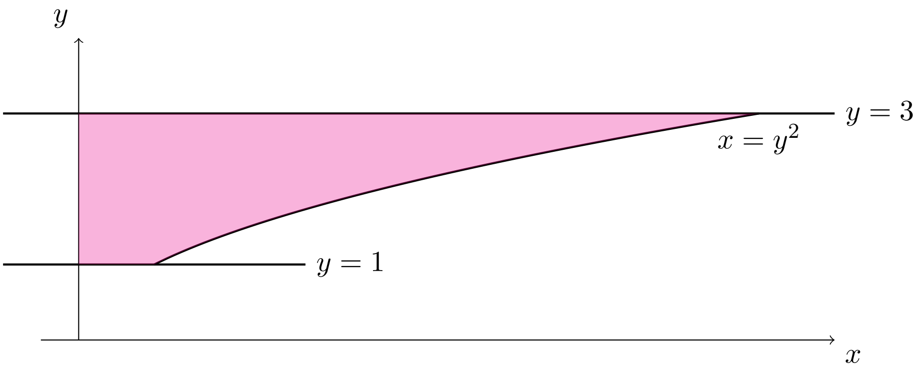
$$\iint_{\mathrm{D}} xy\mathrm{d}x\mathrm{d}y = \int_{-1}^{1}\left(\int_{-x}^{1} xy\mathrm{d}y\right)\mathrm{d}x + \int_{1}^{3}\left(\int_{-1}^{-x+2} xy\mathrm{d}y\right)\mathrm{d}x$$
$$= \int_{-1}^{1}\left(\frac{1}{2}x - \frac{1}{2}x^3\right)\mathrm{d}x + \int_{1}^{3}\left(\frac{1}{2}x^3 - 2x^2 + \frac{3}{2}x\right)\mathrm{d}x$$
$$= \left[\frac{1}{4}x^2 - \frac{1}{8}x^4\right]_{-1}^{1} + \left[\frac{1}{8}x^4 - \frac{2}{3}x^3 + \frac{3}{4}x^2\right]_{1}^{3}$$
$$= -\frac{4}{3}$$

(2)



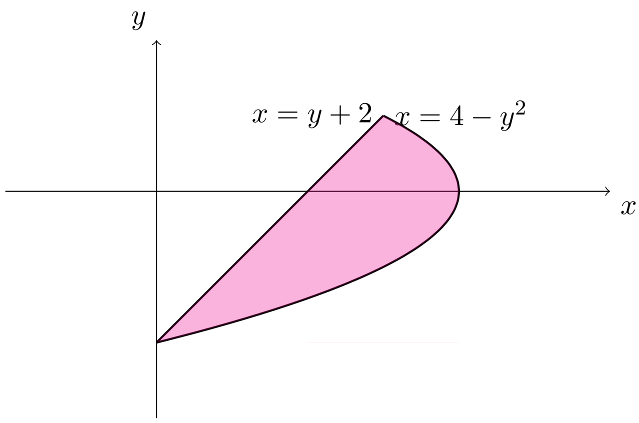
$$\iint_{\mathrm{D}} 2xy\mathrm{d}x\mathrm{d}y = \int_{0}^{1}\left(2x\int_{x^3}^{x^2} y\mathrm{d}y\right)\mathrm{d}x$$
$$= \int_{0}^{1}\left(x^5 - x^7\right)\mathrm{d}x$$
$$= \frac{1}{24}$$

(3)



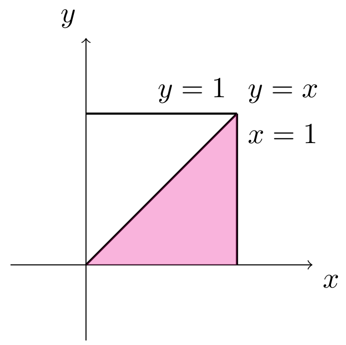
$$
\begin{aligned}
\iint_{\mathrm{D}} e^{\frac{x}{y^2}}\mathrm{d}x\mathrm{d}y &= \int_1^3 \left(\int_0^{y^2} e^{\frac{x}{y^2}}\mathrm{d}x\right)\mathrm{d}y \\
&= \int_1^3 \left(\left[y^2 e^{\frac{x}{y^2}}\right]_0^{y^2}\right)\mathrm{d}y \\
&= \int_1^3 \left((e-1)\,y^2\right)\mathrm{d}y \\
&= \left[\frac{1}{3}(e-1)\,y^3\right]_1^3 \\
&= \frac{26}{3}(e-1)
\end{aligned}
$$

(4)



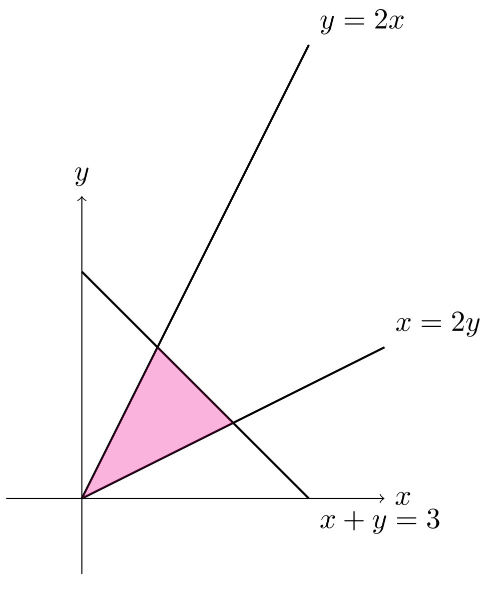
$$
\begin{aligned}
\iint_{\mathrm{D}} x^2 y\mathrm{d}x\mathrm{d}y &= \int_{-2}^{1}\left(y\int_{y+2}^{4-y^2} x^2\mathrm{d}x\right)\mathrm{d}y \\
&= \frac{1}{3}\int_{-2}^{1}\left(-y^7+12y^5-y^4-54y^3-12y^2+56y\right)\mathrm{d}y \\
&= \frac{1}{3}\left[-\frac{1}{8}y^8+2y^6-\frac{1}{5}y^5-\frac{27}{2}y^4-4y^3+28y^2\right]_{-2}^{1} \\
&= -\frac{243}{40}
\end{aligned}
$$

(5)



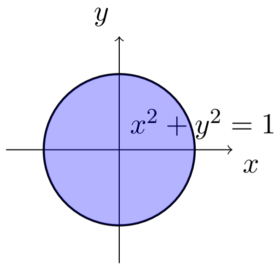
$$\begin{aligned}\iint_D x + y \mathrm{d}x\mathrm{d}y &= \int_0^1 \left(\int_0^x (x+y)\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^1 \left(\left[xy + \frac{1}{2}y^2\right]_0^x\right)\mathrm{d}x \\ &= \frac{3}{2}\int_0^1 x^2 \mathrm{d}x \\ &= \frac{1}{2}\left[x^3\right]_0^1 \\ &= \frac{1}{2}\end{aligned}$$

(6)



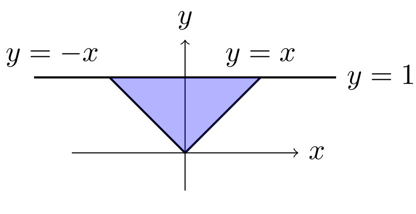
$$\begin{aligned}\iint_{\mathrm{D}}(y-x)\,\mathrm{d}x\mathrm{d}y &= \int_0^1\left(\int_{\frac{1}{2}x}^{2x}(y-x)\,\mathrm{d}y\right)\mathrm{d}x + \int_1^2\left(\int_{\frac{1}{2}x}^{-x+3}(y-x)\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^1\left(\left[\frac{1}{2}y^2-xy\right]_{\frac{1}{2}x}^{2x}\right)\mathrm{d}x + \int_1^2\left(\left[\frac{1}{2}y^2-xy\right]_{\frac{1}{2}x}^{-x+3}\right)\mathrm{d}x \\ &= \frac{3}{8}\int_0^1 x^2\mathrm{d}x + \int_1^2\left(\frac{15}{8}x^2-6x+\frac{9}{2}\right)\mathrm{d}x \\ &= \frac{1}{8}\left[x^3\right]_0^1 + \left[\frac{5}{8}x^3-3x^2+\frac{9}{2}x\right]_1^2 \\ &= 0\end{aligned}$$

(7)



$$\begin{aligned}\iint_{\mathrm{D}}\left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y &= \int_{-1}^1\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(x^2+y^2\right)\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{-1}^1\left(\left[\frac{1}{3}y^3+x^2y\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\right)\mathrm{d}x \\ &= 2\int_{-1}^1\left(\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}+x^2\left(1-x^2\right)^{\frac{1}{2}}\right)\mathrm{d}x \\ &= \frac{1}{6}\left[x\sqrt{1-x^2}\left(2x^2+1\right)-6\arctan\frac{\sqrt{1-x^2}}{1+x}\right]_{-1}^1 \\ &= \frac{\pi}{2}\end{aligned}$$

(8)



$$\begin{aligned}\iint_{\mathrm{D}} ye^x\mathrm{d}x\mathrm{d}y &= \int_0^1\left(\int_{-y}^{y} ye^x\mathrm{d}x\right)\mathrm{d}y \\ &= \int_0^1 y\left(e^y-e^{-y}\right)\mathrm{d}y \\ &= \frac{2}{e}\end{aligned}$$

A8.2

(1)

$$\int_0^1 \int_t^2 f(t,s)\,\mathrm{d}s\mathrm{d}t = \int_0^2 \int_0^s f(t,s)\,\mathrm{d}t\mathrm{d}t$$

(2)

$$\begin{aligned}\int_0^1 \int_t^{1-t} f(t,s)\,\mathrm{d}s\mathrm{d}t &= \int_0^{\frac{1}{2}} \int_t^{1-t} f(t,s)\,\mathrm{d}s\mathrm{d}t \\ &= \int_0^{\frac{1}{2}} \int_0^s f(t,s)\,\mathrm{d}t\mathrm{d}s + \int_{\frac{1}{2}}^1 \int_0^{1-s} f(t,s)\,\mathrm{d}t\mathrm{d}s\end{aligned}$$

(3)

$$\int_0^1 \int_t^{2t} f(t,s)\,\mathrm{d}s\mathrm{d}t = \int_0^1 \int_{\frac{1}{2}s}^s f(t,s)\,\mathrm{d}t\mathrm{d}s + \int_1^2 \int_{\frac{1}{2}s}^1 f(t,s)\,\mathrm{d}t\mathrm{d}s$$

(4)

$$\int_0^1 \int_{t^2}^t f(t,s)\,\mathrm{d}s\mathrm{d}t = \int_0^1 \int_s^{\sqrt{s}} f(t,s)\,\mathrm{d}t\mathrm{d}s$$

(5)

$$\int_0^{\log 2} \int_{e^{-t}}^{e^t} f(t,s)\,\mathrm{d}s\mathrm{d}t = \int_{\frac{1}{2}}^1 \int_{-\log s}^{\log 2} f(t,s)\,\mathrm{d}t\mathrm{d}s + \int_1^2 \int_{\log s}^{\log 2} f(t,s)\,\mathrm{d}t\mathrm{d}s$$

(6)

$$\int_0^4 \int_{\sqrt{t}}^{\sqrt{4-t}+2} f(t,s)\,\mathrm{d}s\mathrm{d}t = \int_0^2 \int_0^{s^2} f(t,s)\,\mathrm{d}t\mathrm{d}s + \int_2^4 \int_0^{4-(s-2)^2} f(t,s)\,\mathrm{d}t\mathrm{d}s$$

A8.3

(1)

$$\begin{aligned}\iiint_{\mathrm{D}} xyz\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1 \int_0^x \int_0^y xyz\mathrm{d}z\mathrm{d}y\mathrm{d}x \\ &= \int_0^1 \int_0^x \frac{1}{2}xy^3\mathrm{d}y\mathrm{d}x \\ &= \frac{1}{8}\int_0^1 x^5\mathrm{d}x \\ &= \frac{1}{48}\end{aligned}$$

(2)

$$\begin{aligned}\iiint_{\mathrm{D}}(x+y+z)\,xyz\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1\left(\int_0^{1-x}\left(\int_0^{1-x-y}(x+y+z)\,xyz\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x\\ &= \int_0^1\left(\int_0^{1-x}\left(\frac{1}{6}xy^4+\frac{1}{2}x^2y^3+\left(\frac{1}{2}x^3-\frac{1}{2}x\right)y^2+\left(\frac{1}{6}x^4-\frac{1}{2}x^2+\frac{1}{3}x\right)y\right)\mathrm{d}y\right)\mathrm{d}\\ &= \frac{1}{120}\int_0^1 x\,(x-1)^4\,(x+4)\,\mathrm{d}x\\ &= \frac{1}{840}\end{aligned}$$

[1]

(3)

$$\begin{aligned}\iiint_{\mathrm{D}}\left(x^2+y^2+z^2\right)xyz\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}}\left(x^2+y^2+z^2\right)xyz\mathrm{d}z\mathrm{d}y\mathrm{d}x\\ &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}0\mathrm{d}y\mathrm{d}x\\ &= 0\end{aligned}$$

(4)

$$\begin{aligned}\iiint_{\mathrm{D}}\sin(x+y+z)\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^{\frac{\pi}{2}}\int_0^{x}\int_0^{x+y}\sin(x+y+z)\mathrm{d}z\mathrm{d}y\mathrm{d}x\\ &= \int_0^{\frac{\pi}{2}}\int_0^{x}\left(\cos(x+y)-\cos(2(x+y))\right)\mathrm{d}y\mathrm{d}x\\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(-2\sin x+3\sin 2x-\sin 4x\right)\mathrm{d}x\\ &= \frac{1}{2}\end{aligned}$$

(5)

$$\begin{aligned}\iiint_{\mathrm{D}}y^2\mathrm{d}x\mathrm{d}y\mathrm{d}z &= 8\int_0^1\int_0^{1-x}\int_0^{1-x-y}y^2\mathrm{d}z\mathrm{d}y\mathrm{d}x\\ &= 8\int_0^1\int_0^{1-x}\left(-y^3-xy^2+y^2\right)\mathrm{d}y\mathrm{d}x\\ &= \frac{2}{3}\int_0^1(x-1)^4\,\mathrm{d}x\\ &= \frac{2}{15}\end{aligned}$$

B8.4

(1)

n のとき成立すると仮定する

$$\int_0^1 \int_0^{x_1} \cdots \int_0^{x_{n-1}} \mathrm{d}x_n \cdots \mathrm{d}x_2 \mathrm{d}x_1 = \frac{1}{n!}$$

$n+1$ のときでは

$$\begin{aligned}
\text{LHS} &= \int_0^1 \int_0^{x_1} \cdots \int_0^{x_{n-1}} \int_0^{x_n} \mathrm{d}x_{n+1} \mathrm{d}x_n \cdots \mathrm{d}x_2 \mathrm{d}x_1 \\
&= \int_0^1 \int_0^{x_1} \cdots \int_0^{x_{n-1}} x_n \mathrm{d}x_n \cdots \mathrm{d}x_2 \mathrm{d}x_1 \\
&= \frac{1}{2} \int_0^1 \int_0^{x_1} \cdots \int_0^{x_{n-2}} x_{n-1}^2 \mathrm{d}x_{n-1} \cdots \mathrm{d}x_2 \mathrm{d}x_1 \\
&= \vdots \\
&= \frac{1}{n!} \int_0^1 x_1^n \mathrm{d}x_1 \\
&= \frac{1}{(n+1)!}
\end{aligned}$$

帰納法より証明終わり □

(2)

$$\begin{aligned}
\int_0^x \left(\cdots \int_0^{x_2} f(x_1) \mathrm{d}x_1 \right) \mathrm{d}x &= \int_0^x \left(\cdots \int_0^{x_3} f_1(x_2) \mathrm{d}x_2 \right) \mathrm{d}x \\
&= \vdots \\
&= \int_0^x f_{n-1}(x_n) \mathrm{d}x_n \\
&= f_n(x)
\end{aligned}$$

また、二つ目の等号について
n のとき

$$f_n(x) = \frac{1}{(n-1)!} \int_0^x f(t) (x-t)^{n-1} \mathrm{d}t$$

$n+1$ のとき

$$\begin{aligned}
f_{n+1}(x) &= \int_0^x f_n(t) \mathrm{d}t \\
&= \int_0^x \left(\frac{1}{(n-1)!} \int_0^t f(s) (t-s)^{n-1} \mathrm{d}s \right) \mathrm{d}t \\
&\overset{Fubini}{=} \frac{1}{(n-1)!} \int_0^t \left(\int_0^x f(s) (t-s)^{n-1} \mathrm{d}t \right) \mathrm{d}s \\
&= \frac{1}{n!} \int_0^t f(s) (x-s)^n \mathrm{d}s \\
&\overset{3^{rd}}{=} \frac{1}{n!} \int_0^x f(t) (x-t)^n \mathrm{d}t
\end{aligned}$$

帰納法より証明おわり □

(3)

$$\begin{aligned}\int_A \mathrm{d}x &\overset{(1)}{=} \int_{-a}^{a} \cdots \int_0^{x_{N-1}} \mathrm{d}x_N \cdots \mathrm{d}x_2 \mathrm{d}x_1 \\ &\overset{(2)}{=} 2^N \cdot \frac{1}{N!} \cdot a^N \\ &= \frac{(2a)^N}{N!}\end{aligned}$$

B8.5

(1)

$|A| = \int_I \chi_A(x)\, \mathrm{d}x$
$\chi_A(x) \geq 0$ から、$|A| \geq \int_I 0 \mathrm{d}x = 0$
一方
$\chi_\emptyset = 0 \Longrightarrow |\emptyset| = \int_I 0 \mathrm{d}x = 0$

(2)

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B| \\ &\overset{(1)}{=} |A| + |B|\end{aligned}$$

B8.6

(1)

$\{a\} \subset [a, a)$
$|[a, a)| = 0$
$(I_k)_{1 \leq k \leq n} = [a, a)$ とすれば
証明終わり □

(2)

$N_j \subset \bigcup_{j,n=1}^{\infty} I_{j,n} \Longrightarrow \bigcup_{j=1}^{\infty} N_j \subset \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} I_{j,n}\ \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} I_{j,n} < \bigcup_{j=1}^{\infty} \epsilon \leq \epsilon_0$

(3)

$N' \subset N \subset \mathbb{R}$ かつ N は *Lebesgue* 零集合であるから
$N' \subset N \subset \bigcup_{n=1}^{\infty} I_n, \sum_{n=1}^{\infty} |I_n| < \epsilon$
よって集合族 $\{I_k\}_{1 \leq k \leq n}$ は $N' \subset \bigcup_{n=1}^{\infty} I_n, \sum_{n=1}^{\infty} |I_n| < \epsilon$ をみたす集合族である
すなわち、N' も *Lebesgue* 零集合である

(4)

$\mathbb{P} := [0,1] \cap \mathbb{Q}$ とする
$\overline{\mu}(\mathbb{P}) = \overline{\int} \chi_{\mathbb{P}} \mathrm{d}x = 1 \neq 0 = \underline{\int} \chi_{\mathbb{P}} \mathrm{d}x = \underline{\mu}(\mathbb{P})$
$\Longrightarrow \mathbb{P}$ は不可測である
(1) と (2) より、有理数集合 $\mathbb{Q}$ は $Lebesgue$ 零集合である（無限個の一点（有理点）集合の和集合も $Lebesgue$ 零集合）
だから、有理集合 $\mathbb{Q}$ の部分集合 $[0,1] \cap \mathbb{Q}$ も $Lebesgue$ 零集合

参考文献

[1] B.P. Demidovich. Problems in Mathematical Analysis. Russian Monographs and Texts on Advanced Mathematics and Physics, 1964.