§9

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A9.1

(1)

$\begin{pmatrix} u \\ v \end{pmatrix} = \Phi^{-1}(x, y) = \begin{pmatrix} \frac{1}{2}(x+y) \\ \frac{1}{2}(x-y) \end{pmatrix}$ で

$\mathrm{J}\Phi^{-1} = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2} \Longrightarrow \frac{\partial(x, y)}{\partial(u, v)} = -2$

$$\begin{aligned}
\iint_A \sin(x+y)\,\mathrm{d}x\mathrm{d}y &= \int_0^{\frac{\pi}{2}} \int_{-u}^{u} \sin 2u \left| \frac{1}{\mathrm{J}\Phi^{-1}} \right| \mathrm{d}v\mathrm{d}u \\
&= 2\int_0^{\frac{\pi}{2}} \int_{-u}^{u} \sin 2u\mathrm{d}v\mathrm{d}u \\
&= 4\int_0^{\frac{\pi}{2}} u \sin 2u\mathrm{d}u \\
&= 4\left\{ \left[-\frac{1}{2}u\cos 2u \right]_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2u\mathrm{d}u \right\} \\
&= \pi
\end{aligned}$$

(2)

$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi(u, v) = \begin{pmatrix} uv \\ v \end{pmatrix}$ で

$\mathrm{J}\Phi = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$

$$\begin{aligned}
\iint_A x^2 y\mathrm{d}x\mathrm{d}y &= \int_0^1 \int_0^1 u^2 v^3 \cdot v\mathrm{d}v\mathrm{d}u \\
&= \int_0^1 \int_0^1 u^2 v^4 \mathrm{d}v\mathrm{d}u \\
&= \frac{1}{5}\int_0^1 u^2 \mathrm{d}u \\
&= \frac{1}{15}
\end{aligned}$$

(3)

$\begin{pmatrix} u \\ v \end{pmatrix} = \Phi^{-1}(x,y) = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$ で

$\mathrm{J}\Phi^{-1} = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \Longrightarrow \dfrac{\partial(x,y)}{\partial(u,v)} = -\dfrac{1}{2}$

$$\begin{aligned}
\iint_A (x-y)\, e^{x^2-y^2} \mathrm{d}x\mathrm{d}y &= \frac{1}{2}\int_1^4 \int_0^1 v e^{uv} \mathrm{d}u\mathrm{d}v \\
&= \frac{1}{2}\int_1^4 (e^v - 1)\,\mathrm{d}v \\
&= \frac{1}{2}\left[e^v - v\right]_1^4 \\
&= \frac{1}{2}\left(e^4 - e - 3\right)
\end{aligned}$$

(4)

$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi(u,v) = \begin{pmatrix} u+uv \\ u-uv \end{pmatrix}$

$\mathrm{J}\Phi = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ 1-v & -u \end{vmatrix} = -2u$

$$\begin{aligned}
\iint_A \exp\left(\frac{x-y}{x+y}\right)\mathrm{d}x\mathrm{d}y &= 2\int_1^2 \int_{-1}^1 u \exp\frac{2uv}{2u} \mathrm{d}v\mathrm{d}u \\
&= 2\int_1^2 \int_{-1}^1 u e^v \mathrm{d}v\mathrm{d}u \\
&= 2\left(e - \frac{1}{e}\right)\int_1^2 u\mathrm{d}u \\
&= 3\left(e - \frac{1}{e}\right)
\end{aligned}$$

(5)

$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi(u,v) = \begin{pmatrix} \sqrt{\dfrac{u}{v}} \\ \sqrt{uv} \end{pmatrix}$

$\mathrm{J}\Phi = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{1}{2\sqrt{uv}} & -\dfrac{1}{2v}\sqrt{\dfrac{u}{v}} \\ \dfrac{v}{2\sqrt{uv}} & \dfrac{u}{2\sqrt{uv}} \end{vmatrix} = \dfrac{1}{2v}$

$$\begin{aligned}
\iint_A y^2 \mathrm{d}x\mathrm{d}y &= \frac{1}{2}\int_1^3 \int_1^2 uv \cdot \frac{1}{v} \mathrm{d}v\mathrm{d}u \\
&= \frac{1}{2}\int_1^3 \int_1^2 u\mathrm{d}v\mathrm{d}u \\
&= \frac{1}{2}\int_1^3 u\mathrm{d}u \\
&= \frac{1}{4}\left[u^2\right]_1^3 \\
&= 2
\end{aligned}$$

A9.2

(1)

$$\begin{aligned}\iint_D \left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y &= \int_1^{\sqrt{2}}\int_0^{2\pi} r^2\cdot r\mathrm{d}\theta\mathrm{d}r \\ &= 2\pi\int_1^{\sqrt{2}} r^3\mathrm{d}r \\ &= \frac{\pi}{2}\left[r^4\right]_1^{\sqrt{2}} \\ &= \frac{3}{2}\pi\end{aligned}$$

(2)

$$\begin{aligned}\iint_D \cos\left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y &= \int_0^1\int_0^{\frac{\pi}{2}} r\cos r^2\mathrm{d}\theta\mathrm{d}r \\ &= \frac{\pi}{2}\int_0^1 r\cos r^2\mathrm{d}r \\ &= \frac{\pi}{4}\sin 1\end{aligned}$$

(3)

$$\begin{aligned}\iint_D e^{-\left(x^2+y^2\right)}\mathrm{d}x\mathrm{d}y &= \int_0^2\int_0^{2\pi} re^{-r^2}\mathrm{d}\theta\mathrm{d}r \\ &= 2\pi\int_0^2 re^{-r^2}\mathrm{d}r \\ &= \pi\left(1-\frac{1}{e^4}\right)\end{aligned}$$

(4)

$\begin{pmatrix} x \\ y \end{pmatrix} = \Psi(r,\theta) = \begin{pmatrix} 1+r\cos\theta \\ r\sin\theta \end{pmatrix} \Longrightarrow \mathrm{J}\Psi = r$

$$\begin{aligned}\iint_D xy\mathrm{d}x\mathrm{d}y &= \int_0^\pi\int_0^1 \left(r\sin\theta\cdot(r\cos\theta+1)\cdot r\right)\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^\pi\left(\frac{1}{4}\sin\theta\cos\theta+\frac{\sin\theta}{3}\right)\mathrm{d}\theta \\ &= \frac{2}{3}\end{aligned}$$

(5)

ここの範囲は (4) と同じであるから、同じように $\begin{pmatrix} x \\ y \end{pmatrix} = \Psi(r,\theta) = \begin{pmatrix} 1+r\cos\theta \\ r\sin\theta \end{pmatrix}$ を考えよう

$$\begin{aligned}
\iint_D x^2y\mathrm{d}x\mathrm{d}y &= \int_0^{\pi}\int_0^1 \left(r\sin\theta\cdot(r\cos\theta+1)^2\cdot r\right)\mathrm{d}r\mathrm{d}\theta \\
&= \frac{1}{30}\int_0^{\pi}\sin\theta\,(3\cos 2\theta+15\cos\theta+13)\,\mathrm{d}\theta \\
&= \frac{4}{5}
\end{aligned}$$

(6)

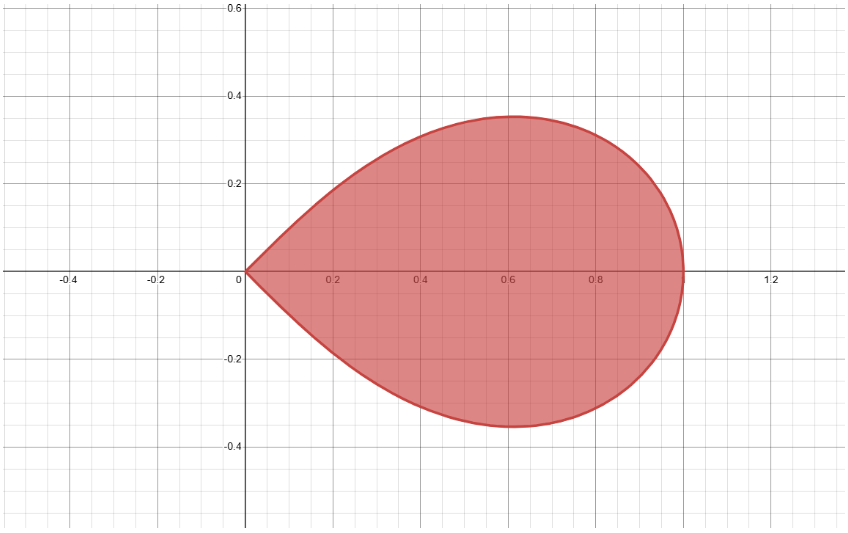

Figure 1:

計算してみると、境界の極座標にした方程式は $r=\sqrt{\cos 2\theta}, \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ である

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Psi(r,\theta) = \begin{pmatrix} r\sqrt{\cos 2\theta}\cos\theta \\ r\sqrt{\cos 2\theta}\sin\theta \end{pmatrix}$$

$$\mathrm{J}\Psi = \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \cos\theta\sqrt{\cos 2\theta} & -\dfrac{r\sin 3\theta}{\sqrt{\cos 2\theta}} \\ \sin\theta\sqrt{\cos 2\theta} & \dfrac{r\cos 3\theta}{\sqrt{\cos 2\theta}} \end{pmatrix} = r\cos 2\theta$$

$$\begin{aligned}
\iint_D \frac{1}{\sqrt{1+x^2+y^2}}\mathrm{d}x\mathrm{d}y &= \int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{(1+r^2)^2}\cdot r\cos 2\theta\mathrm{d}\theta\mathrm{d}r \\
&= \int_0^1 \frac{r}{(1+r^2)^2}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos 2\theta\mathrm{d}\theta\right)\mathrm{d}r \\
&= 0
\end{aligned}$$

(7)

$$
\begin{aligned}
\iint_D \frac{1}{\sqrt{1+x^2+y^2}}\mathrm{d}x\mathrm{d}y &= \int_1^{\sqrt{2}}\int_0^{2\pi}\frac{r}{\sqrt{1+r^2}}\mathrm{d}\theta\mathrm{d}r \\
&= 2\pi\int_1^{\sqrt{2}}\frac{r}{\sqrt{1+r^2}}\mathrm{d}r \\
&= 2\pi\left(\sqrt{3}-\sqrt{2}\right)
\end{aligned}
$$

(8)

$$
\begin{aligned}
\iint_D \frac{xy}{\sqrt{x^2+y^2}}\mathrm{d}x\mathrm{d}y &= \int_0^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{r^3\sin 2\theta}{2r}\mathrm{d}\theta\mathrm{d}r \\
&= \frac{1}{2}\int_0^2\left(r\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin 2\theta\mathrm{d}\theta\right)\mathrm{d}r \\
&= 0
\end{aligned}
$$

A9.3

(1)

$$
\begin{aligned}
\iiint_D x^2\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1\int_0^{2\pi}\int_0^{\pi} r^2\cos^2\theta\sin^2\phi\cdot r^2\sin\phi\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}r \\
&= \int_0^1\int_0^{2\pi}\left(r^4\cos^2\theta\int_0^{\pi}\sin^2\phi\mathrm{d}\phi\right)\mathrm{d}\theta\mathrm{d}r \\
&= \frac{\pi}{2}\int_0^1\int_0^{2\pi}\left(r^4\cos^2\theta\right)\mathrm{d}\theta\mathrm{d}r \\
&= \frac{\pi}{2}\int_0^1 r^4\left(\int_0^{2\pi}\cos^2\theta\mathrm{d}\theta\right)\mathrm{d}r \\
&= \frac{\pi^2}{2}\int_0^1 r^4\mathrm{d}r \\
&= \frac{\pi^2}{10}
\end{aligned}
$$

(2)

$$
\begin{aligned}
\iiint_D z^2\sqrt{x^2+y^2+z^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} r^2\cos^2\phi\cdot r\cdot r^2\sin\phi\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}r \\
&= \int_0^2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} r^5\cos^2\phi\sin\phi\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}r \\
&= \frac{1}{3}\int_0^2\int_0^{2\pi} r^5\mathrm{d}\theta\mathrm{d}r \\
&= \frac{2}{3}\pi\int_0^2 r^5\mathrm{d}r \\
&= \frac{64}{9}\pi
\end{aligned}
$$

(3)

$$\begin{aligned}
\iiint_D xyz\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^{\sqrt{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} r^3\sin\theta\cos\theta\sin^2\phi\cos\phi\cdot r^2\sin\phi\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}r \\
&= \int_0^{\sqrt{2}}\int_0^{\frac{\pi}{2}} r^5\sin\theta\cos\theta\left(\int_0^{\frac{\pi}{2}}\sin^3\phi\cos\phi\mathrm{d}\phi\right)\mathrm{d}\theta\mathrm{d}r \\
&= \frac{1}{4}\int_0^{\sqrt{2}}\left(r^5\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\mathrm{d}\theta\right)\mathrm{d}r \\
&= \frac{1}{8}\int_0^{\sqrt{2}} r^5\mathrm{d}r \\
&= \frac{1}{6}
\end{aligned}$$

(4)

$$\begin{aligned}
\iiint_D \frac{1}{\sqrt{x^2+y^2+z^2}}\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{1}{r}\cdot r^2\sin\phi\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}r \\
&= \int_0^1\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} r\sin\phi\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}r \\
&= \int_0^1\int_0^{\frac{\pi}{2}} r\mathrm{d}\theta\mathrm{d}r \\
&= \frac{\pi}{2}\int_0^1 r\mathrm{d}r \\
&= \frac{\pi}{4}
\end{aligned}$$

B9.4

(1)

ここでは直接に「面積」を考えるのは難しいが、n 次元の球の「体積」を考えよう
きれいな証明方法:[1]
$G(n) := \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(-\sum_{k=1}^{n}x_k^2\right)\mathrm{d}x_1\cdots\mathrm{d}x_n = \prod_k\int_{-\infty}^{\infty}e^{-x_k^2}\mathrm{d}x_k = \prod^{n}\sqrt{\pi} = \pi^{\frac{n}{2}}$
もう一つの計算のやり方は、帰納的に N 次元の球の球面 ($N-1$ 次元) の「球殻の表面積」を積分して、もう一つの次元の「半径」について積分すれば N 次元の体積を得られる.
$G(n) = \int_0^{\infty}\mathrm{d}r\int_{S^{n-1}(r)}\exp\left(-r^2\right)\mathrm{d}S^{n-1} = \int_0^{\infty}\mathrm{d}r\cdot\exp\left(-r^2\right)S^{n-1}(r)$
$V_n(r)\propto r^n \Longrightarrow S^{n-1} := n\alpha r^{n-1}$
これで計算すると

$$\begin{aligned}
G(n) &= n\alpha\int_0^{\infty} r^{n-1}e^{-r^2}\mathrm{d}r \\
&= \frac{n}{2}\alpha\int_0^{\infty}\left(r^2\right)^{\frac{n}{2}-1}e^{-r^2}\mathrm{d}r^2 \\
&= \frac{\alpha n}{2}\Gamma\left(\frac{n}{2}\right)
\end{aligned}$$

最初ガウス積分の形と比較すると
$\pi^{\frac{n}{2}} = \frac{\alpha n}{2}\Gamma\left(\frac{n}{2}\right) \Longrightarrow \alpha = \frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}$
$\Longrightarrow S^{n-1}(r) = n \cdot \frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)} \cdot r^{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1}$
ここはもう測度が $\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ であることがわかるが、一応最後まで計算しよう
$V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} r^n$(ここで $\frac{1}{2}n\Gamma\left(\frac{n}{2}\right) = \Gamma\left(\frac{n}{2}+1\right)$ を使う)
きれいというより、ここでガウス積分を考える方法は強い注意力が必要. だが、これを使って
これより強い結論を導くことはできないらしい

一般的な方法
帰納的に考えると
n 次元の球の体積を $C_n r^n$ にすると

$$\begin{aligned} V_{n+1} &= \int_{-r}^{r} C_n \left(r^2 - x^2\right)^{\frac{n}{2}} \mathrm{d}x \\ &\overset{x=r\sin\theta}{=} C_n r^{n+1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n+1}\theta \mathrm{d}\theta \\ &= \sqrt{\pi}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}+1\right)} r^{n+1} \end{aligned}$$

c_1 は長さが $x_1^2 \le r^2$ である線分であるから、$c_1 = 2$
その漸化式で書き直すと、$C_{n+1} = C_n \frac{\sqrt{\pi}\left(\frac{n}{2}\right)!}{\left(\frac{n+1}{2}\right)!}$
だから、$C_n = \frac{\pi^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$
$V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} r^n$
これを微分すると S^{n-1} の球面の測度がわかる

(2)

実際これが (1) の一番目の方法の二番目の計算ことからわかる
左側は n 次元の球「体積」で(なぜなら球対称)、右側は球面の単位面積(測度)から導いた
球面面積、そして球半径は ϕ で、球の体積が得られる

B9.5

(1)

$\begin{pmatrix} u \\ v \end{pmatrix} =: \Phi(x, y) = \begin{pmatrix} x + y \\ x \end{pmatrix}$
$\mathrm{J}\Phi = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$$\begin{aligned} \int_D f(x+y)\,\mathrm{d}x\mathrm{d}y &= \int_0^a \int_0^v f(v) \cdot 1 \mathrm{d}u\mathrm{d}v \\ &= \int_0^a v f(v)\,\mathrm{d}v \\ &= \int_0^a x f(x)\,\mathrm{d}x \end{aligned}$$

(2)

$\begin{pmatrix} u \\ v \end{pmatrix} =: \Psi(x, y) = \begin{pmatrix} ax + by \\ -bx + ay \end{pmatrix}$
$\mathrm{J}\Psi = \begin{vmatrix} a & b \\ -b & a \end{vmatrix} = 1$

$$\begin{aligned} \int_D f(ax+by)\,\mathrm{d}x\mathrm{d}y &= \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} f(u)\,\mathrm{d}v\mathrm{d}u \\ &= 2\int_{-1}^1 f(u)\sqrt{1-u^2}\mathrm{d}u \\ &= 2\int_{-1}^1 f(r)\sqrt{1-r^2}\mathrm{d}r \end{aligned}$$

参考文献

[1] Walter Greiner, Ludwig Neise, and Horst Stöcker. Thermodynamics and Statistical Mechanics. Classical Theoretical Physics. Springer New York, NY, 1 edition, 1995. Original German edition published by Harri Deutsch Verlag, 1987.