2.1

$$\sigma(u,v) = \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \frac{1}{\sqrt{3}} \cos u \end{pmatrix} \tag{1}$$

$$\sigma_u(u,v) = \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ -\frac{1}{\sqrt{3}} \sin u \end{pmatrix} \tag{2}$$

$$\sigma_v(u,v) = \begin{pmatrix} -\sin u \sin v \\ \sin u \cos v \\ 0 \end{pmatrix} \tag{3}$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} \frac{1}{\sqrt{3}} \sin^2 u \cos v \\ \frac{1}{\sqrt{3}} \sin^2 u \sin v \\ \sin u \cos u \end{pmatrix} \tag{4}$$

$$\|\sigma_u \times \sigma_v\| = \sqrt{\frac{1}{3}\sin^4 u \cos^2 v + \frac{1}{3}\sin^4 u \sin^2 v + \sin^2 u \cos^2 u} \tag{5}$$

$$= \sin u \sqrt{\frac{1}{3}\sin^2 u + \cos^2 u} \tag{6}$$

$$= \frac{1}{\sqrt{3}} \sin u \sqrt{2\cos^2 u + 1} \tag{7}$$

よって、T の曲面積は

$$Area(T) = \iint_{\left(\frac{\pi}{2},\pi\right)\times(0,2\pi)} \|\sigma_u \times \sigma_v\| \, \mathrm{d}u\mathrm{d}v \tag{8}$$

$$= \iint_{\left(\frac{\pi}{2},\pi\right)\times(0,2\pi)} \frac{1}{\sqrt{3}} \sin u \sqrt{2\cos^2 u + 1} \mathrm{d}u\mathrm{d}v \tag{9}$$

$$= \frac{2}{\sqrt{3}}\pi \int_{\frac{\pi}{2}}^{\pi} \sin u \sqrt{2\cos^2 u + 1} \mathrm{d}u \tag{10}$$

$$= \frac{2\pi}{\sqrt{3}} \int_{-1}^{0} \sqrt{2s^2 + 1} \mathrm{d}s \tag{11}$$

$$= \frac{\sqrt{6}}{3}\pi \int_{-\arctan\sqrt{2}}^{0} \frac{1}{\cos^3 t} \mathrm{d}t \tag{12}$$

ここでまず、$\int \frac{1}{\cos^3 t}\mathrm{d}t$ を考える

$$I = \int \frac{1}{\cos^3 t}\mathrm{d}t \tag{13}$$

$$= \tan t \frac{1}{\cos t} - \int \tan^2 t \frac{1}{\cos t}\mathrm{d}t \tag{14}$$

$$= \tan t \frac{1}{\cos t} - \int \left(\frac{1}{\cos^2 t} - 1\right)\frac{1}{\cos t}\mathrm{d}t \tag{15}$$

$$= \tan t \frac{1}{\cos t} - \int \frac{1}{\cos^3 t}\mathrm{d}t + \int \frac{1}{\cos t}\mathrm{d}t \tag{16}$$

$$= \tan t \frac{1}{\cos t} - \int \frac{1}{\cos^3 t}\mathrm{d}t + \log\left|\tan x + \frac{1}{\cos x}\right| \tag{17}$$

よって、$2\int \frac{1}{\cos^3 t}\mathrm{d}t = \frac{\sin t}{\cos^2 t} + \log\left|\tan t + \frac{1}{\cos t}\right|$

$\iff \int \frac{1}{\cos^3 t}\mathrm{d}t = \frac{1}{2}\left(\frac{\sin t}{\cos^2 t} + \log\left|\tan t + \frac{1}{\cos t}\right|\right)$

$$
\begin{aligned}
Area\,(T) &= \frac{\sqrt{6}}{3}\pi \int_{-\arctan\sqrt{2}}^{0} \frac{1}{\cos^3 t}\mathrm{d}t && (18)\\
&= \frac{\sqrt{6}}{3}\pi \left[\frac{1}{2}\left(\frac{\sin t}{\cos^2 t} + \log\left|\tan t + \frac{1}{\cos t}\right|\right)\right]_{-\arctan\sqrt{2}}^{0} && (19)\\
&= \frac{\sqrt{6}}{3}\pi \left(0 - \frac{1}{2}\left(\frac{\sin\left(-\arctan\sqrt{2}\right)}{\cos^2\left(-\arctan\sqrt{2}\right)} + \log\left|\tan\left(-\arctan\sqrt{2}\right) + \frac{1}{\cos\left(-\arctan\sqrt{2}\right)}\right|\right)\right) && (20)\\
&= \frac{\sqrt{6}}{3}\pi \left(-\frac{1}{2}\left(-\sqrt{6} + \log\left|-\sqrt{2} + \sqrt{3}\right|\right)\right) && (21)\\
&= \pi - \frac{\sqrt{6}}{6}\pi \log\left(\sqrt{3} - \sqrt{2}\right) && (22)
\end{aligned}
$$

2.2

$$
\begin{aligned}
\sigma\,(u, v) &= \begin{pmatrix} u \\ v \\ \sqrt{u^2 + v^2} \end{pmatrix} && (23)\\
\sigma_u &= \begin{pmatrix} 1 \\ 0 \\ \frac{u}{\sqrt{u^2+v^2}} \end{pmatrix} && (24)\\
\sigma_v &= \begin{pmatrix} 0 \\ 1 \\ \frac{v}{\sqrt{u^2+v^2}} \end{pmatrix} && (25)\\
\sigma_u \times \sigma_v &= \begin{pmatrix} -\dfrac{u}{\sqrt{u^2 + v^2}} \\ -\dfrac{v}{\sqrt{u^2 + v^2}} \\ 1 \end{pmatrix} && (26)\\
\|\sigma_u \times \sigma_v\| &= \sqrt{2} && (27)
\end{aligned}
$$

(1)

$$
\begin{aligned}
v_1 &= \begin{pmatrix} x^2 + y - 4 \\ 3xy \\ 2xz + z^2 \end{pmatrix} && (28)\\
\nabla \times v_1 &= \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} x^2 + y - 4 \\ 3xy \\ 2xz + z^2 \end{pmatrix} && (29)\\
&= \begin{pmatrix} 0 \\ -2z \\ 3y - 1 \end{pmatrix} && (30)
\end{aligned}
$$

(2)

$$\iint_{\sigma(\overline{\Omega})} \nabla \times v_1 \mathrm{d}A = \iint_{\sigma(\overline{\Omega})} \begin{pmatrix} 0 \\ -2\sqrt{u^2+v^2} \\ 3v-1 \end{pmatrix} \cdot \begin{pmatrix} -\frac{u}{\sqrt{u^2+v^2}} \\ -\frac{v}{\sqrt{u^2+v^2}} \\ 1 \end{pmatrix} \mathrm{d}u\mathrm{d}v \tag{31}$$

$$= \iint_{\sigma(\overline{\Omega})} (5v-1)\,\mathrm{d}u\mathrm{d}v \tag{32}$$

$$= \int_0^{2\pi}\int_2^3 \left(5a^2\sin t - a\right)\mathrm{d}a\mathrm{d}t \tag{33}$$

$$= \int_0^{2\pi}\left[\frac{5}{3}a^3\sin t - \frac{1}{2}a^2\right]_2^3 \mathrm{d}t \tag{34}$$

$$= \int_0^{2\pi}\left(\frac{95}{3}\sin t - \frac{5}{2}\right)\mathrm{d}t \tag{35}$$

$$= \left[-\frac{95}{3}\cos t - \frac{5}{2}t\right]_0^{2\pi} \tag{36}$$

$$= -5\pi \tag{37}$$

一方 $C_1(t) = \begin{pmatrix} 3\cos t \\ 3\sin t \\ 3 \end{pmatrix}, C_2(t) = \begin{pmatrix} 2\cos t \\ -2\sin t \\ 2 \end{pmatrix}$ とすると

$C_1'(t) = \begin{pmatrix} -3\sin t \\ 3\cos t \\ 0 \end{pmatrix}, C_2'(t) = \begin{pmatrix} -2\sin t \\ -2\cos t \\ 0 \end{pmatrix}$

$$\int_{C_1} v_1 \cdot \mathrm{d}x = \int_0^{2\pi} \begin{pmatrix} 9\cos^2 t + 3\sin t - 4 \\ 27\sin t\cos t \\ 18\cos t + 9 \end{pmatrix} \cdot \begin{pmatrix} -3\sin t \\ 3\cos t \\ 0 \end{pmatrix} \mathrm{d}t \tag{38}$$

$$= \int_0^{2\pi}\left(-27\sin t\cos^2 t - 9\sin^2 t + 12\sin t + 81\sin t\cos^2 t\right)\mathrm{d}t \tag{39}$$

$$= \int_0^{2\pi}\left(54\sin t\cos^2 t - 9\sin^2 t + 12\sin t\right)\mathrm{d}t \tag{40}$$

$$= -9\pi \tag{41}$$

$$\int_{C_2} v_1 \cdot \mathrm{d}x = \int_0^{2\pi} \begin{pmatrix} 4\cos^2 t - 2\sin t - 4 \\ -12\sin t\cos t \\ 8\cos t + 4 \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ -2\cos t \\ 0 \end{pmatrix} \mathrm{d}t \tag{42}$$

$$= \int_0^{2\pi}\left(16\sin t\cos^2 t + 4\sin^2 t + 8\sin t\right)\mathrm{d}t \tag{43}$$

$$= 4\int_0^{2\pi}\left(4\sin t\cos^2 t + \sin^2 t + 2\sin t\right)\mathrm{d}t \tag{44}$$

$$= 4\pi \tag{45}$$

$$\int_{\sigma(\partial\Omega)} v_1 \cdot \mathrm{d}x = \int_{C_1} v_1 \cdot \mathrm{d}x + \int_{C_2} v_1 \cdot \mathrm{d}x = -5\pi$$

(3)

$$v_2 = \begin{pmatrix} 2z - x^2 \\ -2xy - y \\ 2y \end{pmatrix} \tag{46}$$

$$\nabla \times v_2 = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} 2z - x^2 \\ -2xy - y \\ 2y \end{pmatrix} \tag{47}$$

$$= \begin{pmatrix} 2 \\ 2 \\ -2y \end{pmatrix} \tag{48}$$

(4)

$$\iint_{\sigma(\overline{\Omega})} \nabla \times v_2 \mathrm{d}A = \iint_{\sigma(\overline{\Omega})} \begin{pmatrix} 2 \\ 2 \\ -2v \end{pmatrix} \cdot \begin{pmatrix} -\frac{u}{\sqrt{u^2+v^2}} \\ -\frac{v}{\sqrt{u^2+v^2}} \\ 1 \end{pmatrix} \mathrm{d}u\mathrm{d}v \tag{49}$$

$$= -2 \iint_{\sigma(\overline{\Omega})} \left(\frac{u}{\sqrt{u^2+v^2}} + \frac{v}{\sqrt{u^2+v^2}} + v \right) \mathrm{d}u\mathrm{d}v \tag{50}$$

$$= -2 \int_0^{2\pi} \int_2^3 \left(a\left(a+1\right) \sin\theta + a\cos\theta \right) \mathrm{d}a\mathrm{d}\theta \tag{51}$$

$$= -2 \int_0^{2\pi} \left(\frac{23}{6} \sin\theta + \frac{5}{2} \cos\theta \right) \mathrm{d}\theta \tag{52}$$

$$= 0 \tag{53}$$

一方

$$\int_{C_1} v_2 \cdot \mathrm{d}x = \int_0^{2\pi} \begin{pmatrix} 6 - 9\cos^2 t \\ -18\sin t\cos t - 3\sin t \\ 6\sin t \end{pmatrix} \cdot \begin{pmatrix} -3\sin t \\ 3\cos t \\ 0 \end{pmatrix} \mathrm{d}t \tag{54}$$

$$= -9 \int_0^{2\pi} \left(3\sin t\cos^2 t + 2\sin t + \sin t\cos t \right) \mathrm{d}t \tag{55}$$

$$= 0 \tag{56}$$

$$\int_{C_2} v_2 \cdot \mathrm{d}x = \int_0^{2\pi} \begin{pmatrix} 4 - 4\cos^2 t \\ 8\sin t\cos t + 2\sin t \\ -4\sin t \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ -2\cos t \\ 0 \end{pmatrix} \mathrm{d}t \tag{57}$$

$$= -4 \int_0^{2\pi} \left(2\sin t\cos^2 t + 2\sin t + \sin t\cos t \right) \mathrm{d}t \tag{58}$$

$$= 0 \tag{59}$$

よって、$\int_{\sigma(\partial\Omega)} v_2 \cdot \mathrm{d}x = \int_{C_1} v_2 \cdot \mathrm{d}x + \int_{C_2} v_2 \cdot \mathrm{d}x = 0$