4.1

(1)

$$\partial V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| \frac{x^2}{4} + y^2 + z^2 = 1 \right\}$$

(2)

$f\left(x, y, z\right) = \dfrac{x^2}{4} + y^2 + z^2, \nabla f = \begin{pmatrix} \frac{x}{2} \\ 2y \\ 2z \end{pmatrix}$

$$\begin{aligned} \mathbf{n} &= \frac{\nabla f}{\|\nabla f\|} && (1) \\ &= \frac{1}{\sqrt{\frac{x^2}{4} + 4y^2 + 4z^2}} \begin{pmatrix} \frac{x}{2} \\ 2y \\ 2z \end{pmatrix} && (2) \end{aligned}$$

(3)

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} && (3) \\ &= 1 + 1 + 1 = 3 && (4) \end{aligned}$$

(4)

$$\begin{aligned} \iiint_{\overline{V}} \nabla \cdot \mathbf{v} \mathrm{d}x\mathrm{d}y\mathrm{d}z &= \iiint_{\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| \frac{x^2}{4} + y^2 + z^2 \leq 1 \right\}} 3\mathrm{d}x\mathrm{d}y\mathrm{d}z && (5) \\ &= 3 \iiint_{\left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathbb{R}^3 \middle| u^2 + v^2 + w^2 \leq 1 \right\}} 2\mathrm{d}u\mathrm{d}v\mathrm{d}w && (6) \\ &= 6 \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} \mathrm{d}w\mathrm{d}v\mathrm{d}u && (7) \\ &= 6 \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 2\sqrt{1 - u^2 - v^2} \mathrm{d}v\mathrm{d}u && (8) \\ &= 12 \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 2\sqrt{1 - u^2 - v^2} \mathrm{d}v\mathrm{d}u && (9) \\ &= 12 \int_{-1}^{1} \left(-\frac{1}{2}\pi \left(u^2 - 1\right) \right) \mathrm{d}u && (10) \\ &= 6\pi \int_{-1}^{1} \left(1 - u^2\right) \mathrm{d}u && (11) \\ &= 8\pi && (12) \end{aligned}$$

ここで $\sigma(u,v)=\begin{pmatrix}2\sin u\cos v\\ \sin u\sin v\\ \cos u\end{pmatrix}$ とする.

$\sigma_u=\begin{pmatrix}2\cos u\cos v\\ \cos u\sin v\\ -\sin u\end{pmatrix}, \sigma_v=\begin{pmatrix}-2\sin u\sin v\\ \sin u\cos v\\ 0\end{pmatrix}$ であるから

$$\sigma_u\times\sigma_v=\begin{pmatrix}2\cos u\cos v\\ \cos u\sin v\\ -\sin u\end{pmatrix}\times\begin{pmatrix}-2\sin u\sin v\\ \sin u\cos v\\ 0\end{pmatrix} \tag{13}$$
$$=\begin{pmatrix}\sin^2 u\cos v\\ 2\sin^2 u\sin v\\ 2\sin u\cos u\end{pmatrix} \tag{14}$$

で

$$\iint_{\partial V}\mathbf{v}\cdot\mathrm{d}\mathbf{A}=\iint_{[0,\pi]\times[0,2\pi]}\begin{pmatrix}2\sin u\cos v\\ \sin u\sin v\\ \cos u\end{pmatrix}\cdot\begin{pmatrix}\sin^2 u\cos v\\ 2\sin^2 u\sin v\\ 2\sin u\cos u\end{pmatrix}\mathrm{d}u\mathrm{d}v \tag{15}$$
$$=\iint_{[0,\pi]\times[0,2\pi]}\left(2\sin^3 u+2\sin u\cos^2 u\right)\mathrm{d}u\mathrm{d}v \tag{16}$$
$$=4\pi\int_0^\pi\left(\sin^3 u+\sin u\cos^2 u\right)\mathrm{d}u \tag{17}$$
$$=4\pi\left[-\cos u\right]_0^\pi \tag{18}$$
$$=8\pi \tag{19}$$
$$=\iiint_V\nabla\cdot\mathbf{v}\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{20}$$

Rem. 三重積分 $\iiint_{\left\{\begin{pmatrix}x\\ y\\ z\end{pmatrix}\in\mathbb{R}^3\,\middle|\,\frac{x^2}{4}+y^2+z^2\le 1\right\}}\mathrm{d}x\mathrm{d}y\mathrm{d}z$ に対して、分割方法は二つがある.
$z=h>0$ の平面による断面積に対して h に関して積分した体積の 2 倍　と　第一象限の体積の 8 倍

三重積分の別解の 1

まず $z=h>0$ の平面による断面積 $S(h)$ を考える
断面の方程式は $\frac{x^2}{4}+y^2=1-h^2$ であるから、これは楕円 $\frac{x^2}{4(1-h^2)}+\frac{y^2}{1-h^2}=1$ である
$\begin{cases}x=2\sqrt{1-h^2}\cdot u\\ y=\sqrt{1-h^2}\cdot v\end{cases}$ で変数変換すると、$\det\mathrm{J}=\begin{vmatrix}2\sqrt{1-h^2} & 0\\ 0 & \sqrt{1-h^2}\end{vmatrix}=2\left(1-h^2\right)$
だから $S(h)$ は

$$S(h)=\iint_{u^2+v^2\le 1}1\cdot|\det\mathrm{J}|\,\mathrm{d}u\mathrm{d}v \tag{21}$$
$$=2\left(1-h^2\right)\iint_{u^2+v^2\le 1}\mathrm{d}u\mathrm{d}v \tag{22}$$
$$=2\left(1-h^2\right)\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\mathrm{d}v\mathrm{d}u \tag{23}$$
$$=2\left(1-h^2\right)\pi \tag{24}$$

$$\iiint_{\left\{\left(\begin{array}{c}x\\y\\z\end{array}\right)\in\mathbb{R}^3\middle|\frac{x^2}{4}+y^2+z^2\le1\right\}}\mathrm{d}x\mathrm{d}y\mathrm{d}z = 2\int_0^1 S\left(h\right)\mathrm{d}h \tag{25}$$

$$= 2\int_0^1 2\left(1-h^2\right)\pi\mathrm{d}h \tag{26}$$

$$= 4\pi\int_0^1\left(1-h^2\right)\mathrm{d}h \tag{27}$$

$$= \frac{8}{3}\pi \tag{28}$$

だから、$\iiint_{\overline{V}}\nabla\cdot\mathbf{v}\mathrm{d}x\mathrm{d}y\mathrm{d}z = 3\cdot\frac{8}{3}\pi = 8\pi$

三重積分の別解の 2

$$\iiint_{\left\{\left(\begin{array}{c}x\\y\\z\end{array}\right)\in\mathbb{R}^3\middle|\frac{x^2}{4}+y^2+z^2\le1\right\}}\mathrm{d}x\mathrm{d}y\mathrm{d}z = 8\int_0^1\int_0^{2\sqrt{1-z^2}}\int_0^{\sqrt{1-z^2-\frac{x^2}{4}}}\mathrm{d}y\mathrm{d}x\mathrm{d}z \tag{29}$$

$$= 8\int_0^1\int_0^{2\sqrt{1-z^2}}\sqrt{1-z^2-\frac{x^2}{4}}\mathrm{d}x\mathrm{d}z \tag{30}$$

$$= 4\pi\int_0^1\left(1-z^2\right)\mathrm{d}z \tag{31}$$

$$= \frac{8}{3}\pi \tag{32}$$

だから、$\iiint_{\overline{V}}\nabla\cdot\mathbf{v}\mathrm{d}x\mathrm{d}y\mathrm{d}z = 3\cdot\frac{8}{3}\pi = 8\pi$

4.2

$$\nabla\cdot\mathbf{v} = y + 0 + 2y = 3y \tag{33}$$

であるから

$$\iint_{\partial T} \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \iiint_{\overline{T}} \nabla \cdot \mathbf{v} \mathrm{d}x \mathrm{d}y \mathrm{d}z \tag{34}$$

$$= \iiint_{\left\{ \left(\begin{array}{c} x \\ y \\ z \end{array} \right) \in \mathbb{R}^3 \middle| x+y+z \leq 1, x,y,z \geq 0 \right\}} 3y \mathrm{d}x \mathrm{d}y \mathrm{d}z \tag{35}$$

$$= 3 \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y \mathrm{d}z \mathrm{d}y \mathrm{d}x \tag{36}$$

$$= 3 \int_0^1 \int_0^{1-x} \left(y - xy - y^2 \right) \mathrm{d}y \mathrm{d}x \tag{37}$$

$$= 3 \int_0^1 \left[\frac{1}{2} (1-x) y^2 - \frac{1}{3} y^3 \right]_0^{1-x} \mathrm{d}x \tag{38}$$

$$= 3 \int_0^1 \left(\frac{1}{2} (1-x)^3 - \frac{1}{3} (1-x)^3 \right) \mathrm{d}x \tag{39}$$

$$= \frac{1}{2} \int_0^1 (1-x)^3 \mathrm{d}x \tag{40}$$

$$= \frac{1}{8} \tag{41}$$