5.1

$$\omega = \mathrm{d}x_1 - x_3\mathrm{d}x_2 - x_5\mathrm{d}x_4$$

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}\left(\mathrm{d}x_1\right) - \mathrm{d}\left(x_3\mathrm{d}x_2\right) - \mathrm{d}x_5\left(\mathrm{d}x_4\right) && (1)\\
&= 0 - \mathrm{d}x_3 \wedge \mathrm{d}x_2 - \mathrm{d}x_5 \wedge \mathrm{d}x_4 && (2)\\
&= \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_4 \wedge \mathrm{d}x_5 && (3)\\
\mathrm{d}\omega \wedge \mathrm{d}\omega &= (\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_4 \wedge \mathrm{d}x_5) \wedge (\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_4 \wedge \mathrm{d}x_5) && (4)\\
&= \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_4 \wedge \mathrm{d}x_5 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 && (5)\\
&+ \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_5 + \mathrm{d}x_4 \wedge \mathrm{d}x_5 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_5 && (6)\\
&= \mathrm{d}x_4 \wedge \mathrm{d}x_5 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_5 && (7)\\
&= 2\mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_5 && (8)\\
\omega \wedge \mathrm{d}\omega \wedge \mathrm{d}\omega &= (\mathrm{d}x_1 - x_3\mathrm{d}x_2 - x_5\mathrm{d}x_4) \wedge (\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_4 \wedge \mathrm{d}x_5) \wedge (\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_4 \wedge \mathrm{d}x_5) && (9)\\
&= (\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - x_5\mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 + \mathrm{d}x_1 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_5 - x_3\mathrm{d}x_2 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_5) && (10)\\
&\wedge (\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_4 \wedge \mathrm{d}x_5) && (11)\\
&= \mathrm{d}x_1 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_5 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_5 && (12)\\
&= 2\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_5 && (13)
\end{aligned}$$

5.2

$$\omega = \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}y \wedge \mathrm{d}z \\ \mathrm{d}z \wedge \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix}$$

(1)

$$\mathrm{d}\omega = 3\left(x^2 + y^2 + z^2\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \qquad (14)$$

(2)

$$\begin{cases} \mathrm{d}x = \cos u \cos v \mathrm{d}u - \sin u \sin v \mathrm{d}v \\ \mathrm{d}y = \cos u \sin v \mathrm{d}u + \sin u \cos v \mathrm{d}v \\ \mathrm{d}z = -\sin u \mathrm{d}u \end{cases}$$

$$\begin{aligned}
\mathrm{d}x \wedge \mathrm{d}y &= (\cos u \cos v \mathrm{d}u - \sin u \sin v \mathrm{d}v) \wedge (\cos u \sin v \mathrm{d}u + \sin u \cos v \mathrm{d}v) && (15)\\
&= \sin u \cos u \mathrm{d}u \wedge \mathrm{d}v && (16)\\
\mathrm{d}y \wedge \mathrm{d}z &= (\cos u \sin v \mathrm{d}u + \sin u \cos v \mathrm{d}v) \wedge (-\sin u \mathrm{d}u) && (17)\\
&= \sin^2 u \cos v \mathrm{d}u \wedge \mathrm{d}v && (18)\\
\mathrm{d}z \wedge \mathrm{d}x &= (-\sin u \mathrm{d}u) \wedge (\cos u \cos v \mathrm{d}u - \sin u \sin v \mathrm{d}v) && (19)\\
&= \sin^2 u \sin v \mathrm{d}u \wedge \mathrm{d}v && (20)
\end{aligned}$$

$$\begin{aligned}
\phi^*\omega &= \sin^5 u \cos^4 v \mathrm{d}u \wedge \mathrm{d}v + \sin^5 u \sin^4 v \mathrm{d}u \wedge \mathrm{d}v + \sin u \cos^4 u \mathrm{d}u \wedge \mathrm{d}v && (21)\\
&= \sin u \left(\frac{1}{4}\sin^4 u\,(\cos 4v + 3) + \cos^4 u\right)\mathrm{d}u \wedge \mathrm{d}v && (22)
\end{aligned}$$

(3)

$$\int_{\phi|K} \omega = \iint_{(0,\pi)\times(0,2\pi)} \sin u \left(\frac{1}{4}\sin^4 u\,(\cos 4v + 3) + \cos^4 u\right) \mathrm{d}u\mathrm{d}v \tag{23}$$

$$= \int_0^{\pi} \int_0^{2\pi} \sin u \left(\frac{1}{4}\sin^4 u\,(\cos 4v + 3) + \cos^4 u\right) \mathrm{d}v\mathrm{d}u \tag{24}$$

$$= \int_0^{\pi} \left(2\pi \sin u \cos^4 u + \frac{3}{2}\pi \sin^5 u\right) \mathrm{d}u \tag{25}$$

$$= \frac{12}{5}\pi \tag{26}$$

5.3

$\omega = x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y$ とする

$$\int_{\partial V} (x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y) = \int_{\overline{V}} \mathrm{d}\omega \tag{27}$$

$$= \int_{\overline{V}} 3\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{28}$$

$$= 4\pi r^3 \tag{29}$$