§1

1.1

$$\begin{aligned}\int_C (2x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y &= \int_0^1 \left(2t-t^2\right)\mathrm{d}t + \left(t+t^2\right)2t\mathrm{d}t \\ &= \int_0^1 \left(2t^3+t^2+2t\right)\mathrm{d}t \\ &= \left[\frac{1}{2}t^4+\frac{1}{3}t^3+t^2\right]_0^1 \\ &= \frac{11}{6}\end{aligned}$$

1.2

$$\begin{aligned}\int_C \mathbf{v}\cdot\mathrm{d}\mathbf{x} = &\int_0^1 \begin{pmatrix} 2(a+t)(b+t)c \\ (a+t)^2 c \\ (a+t)^2(b+t)\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}\mathrm{d}t + \int_1^2 \begin{pmatrix} 2(a+1)(b+1)(c-1+t) \\ (a+1)^2(c-1+t) \\ (a+1)^2(b+1)\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\mathrm{d}t \\ &= \int_0^1 \left(2(a+t)(b+t)c+(a+t)^2c\right)\mathrm{d}t + \int_1^2 (a+1)^2(b+1)\,\mathrm{d}t \\ &= c\left(a^2+2a(b+1)+b+1\right)+(a+1)^2(b+1) \\ &= a^2(b+c+1)+(2a+1)(b+1)(c+1)\end{aligned}$$

また、線積分の性質を考えると

$$\begin{aligned}\int_C \mathbf{v}\cdot\mathrm{d}\mathbf{x} &= \int_C \nabla\left(x^2yz\right)\mathrm{d}\mathbf{x} \\ &= (a+1)^2(b+1)c - a^2bc + (a+1)^2(b+1)(c+1) - (a+1)^2(b+1)c \\ &= (a+1)^2(b+1)(c+1) - a^2bc\end{aligned}$$

1.3

$$\mathbf{v} = \begin{pmatrix} y-x \\ 3x+2y\end{pmatrix}, D := \left\{\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2 \middle| 0\le x,y\le 1, y\le x\right\}$$

$$\begin{aligned}\int_C (y-x)\,\mathrm{d}x+(3x+2y)\,\mathrm{d}y &= \iint_D \nabla\times\begin{pmatrix} y-x \\ 3x+2y\end{pmatrix}\mathrm{d}x\mathrm{d}y \\ &= \iint_D 2\mathrm{d}x\mathrm{d}y \\ &= \int_0^1\int_0^x 2\mathrm{d}y\mathrm{d}x \\ &= \int_0^1 2x\mathrm{d}x \\ &= 1\end{aligned}$$

1.4

(i)

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)&=\frac{-x^2+y^2}{\left(x^2+y^2\right)^2}-\frac{-x^2+y^2}{\left(x^2+y^2\right)^2}\\&=0\end{aligned}$$

(ii)

場合分けにして、$\begin{pmatrix}0\\0\end{pmatrix}$ は閉曲線の内部に存在しないとき

$$\begin{aligned}\int_C \frac{-y}{x^2+y^2}\mathrm{d}x+\frac{x}{x^2+y^2}\mathrm{d}y&=\iint_D 0\mathrm{d}x\mathrm{d}y\\&=0\end{aligned}$$

$\begin{pmatrix}0\\0\end{pmatrix}$ が内部に存在するとき、計算便利のため $C=\begin{pmatrix}r\cos\theta\\r\sin\theta\end{pmatrix}$ とする.$C'=\begin{pmatrix}-r\sin\theta\\r\cos\theta\end{pmatrix}$

$$\begin{aligned}\int_C \mathbf{v}\mathrm{d}\mathbf{x}&=\int_0^{2\pi}\begin{pmatrix}\frac{-r\sin\theta}{r^2}\\\frac{r\cos\theta}{r^2}\end{pmatrix}\cdot\begin{pmatrix}-r\sin\theta\\r\cos\theta\end{pmatrix}\mathrm{d}\theta\\&=\int_0^{2\pi}1\mathrm{d}\theta\\&=2\pi\end{aligned}$$

参考文献