10.1

$$
\begin{aligned}
\mathrm{d}\alpha &= \mathrm{d}\left(ze^{x+y}\mathrm{d}x\right) \\
&= \mathrm{d}\left(ze^{x+y}\right)\wedge\mathrm{d}x \\
&= \left(ze^{x+y}\mathrm{d}x + ze^{x+y}\mathrm{d}y + e^{x+y}\mathrm{d}z\right)\wedge\mathrm{d}x \\
&= -ze^{x+y}\mathrm{d}x\wedge\mathrm{d}y - e^{x+y}\mathrm{d}x\wedge\mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{d}\omega = \mathrm{d}\left(2z\left(y^2-x^2\right)\mathrm{d}x\wedge\mathrm{d}y + \left(y^3-z^3\right)\mathrm{d}x\wedge\mathrm{d}z + x^3\mathrm{d}y\wedge\mathrm{d}z\right) \\
&= \mathrm{d}\left(\begin{pmatrix} x^3 \\ z^3-y^3 \\ 2z\left(y^2-x^2\right)\end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}y\wedge\mathrm{d}z \\ \mathrm{d}z\wedge\mathrm{d}x \\ \mathrm{d}x\wedge\mathrm{d}y\end{pmatrix}\right) \\
&= \left(\nabla\cdot\begin{pmatrix} x^3 \\ z^3-y^3 \\ 2z\left(y^2-x^2\right)\end{pmatrix}\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \\
&= \left(3x^2-3y^2+2\left(y^2-x^2\right)\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \\
&= \left(x^2-y^2\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z
\end{aligned}
$$

10.2

(1)

$$
\begin{aligned}
\mathrm{d}\alpha &= \mathrm{d}\left(\begin{pmatrix} axyz \\ bx^2z \\ -3x^2y\end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z\end{pmatrix}\right) \\
&= \left(\nabla\times\begin{pmatrix} axyz \\ bx^2z \\ -3x^2y\end{pmatrix}\right)\cdot\begin{pmatrix}\mathrm{d}y\wedge\mathrm{d}z \\ \mathrm{d}z\wedge\mathrm{d}x \\ \mathrm{d}x\wedge\mathrm{d}y\end{pmatrix} \\
&= \begin{pmatrix} -3x^2-bx^2 \\ axy+6xy \\ 2bxz-axz\end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}y\wedge\mathrm{d}z \\ \mathrm{d}z\wedge\mathrm{d}x \\ \mathrm{d}x\wedge\mathrm{d}y\end{pmatrix} \\
&= 0
\end{aligned}
$$

よって、$\begin{cases} -3x^2-bx^2=0 \\ axy+6xy=0 \\ 2bxz-axz=0 \end{cases} \implies \begin{cases} a=-6 \\ b=-3 \end{cases}$

(2)

$$
\begin{aligned}
\mathrm{d}\beta &= \mathrm{d}\left(\begin{pmatrix} -\left(2xe^y-ce^x\right) \\ \left(5ye^z-be^y\right) \\ \left(3ze^x+ae^z\right)\end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}y\wedge\mathrm{d}z \\ \mathrm{d}z\wedge\mathrm{d}x \\ \mathrm{d}x\wedge\mathrm{d}y\end{pmatrix}\right) \\
&= \left(\nabla\cdot\begin{pmatrix} -\left(2xe^y-ce^x\right) \\ \left(5ye^z-be^y\right) \\ \left(3ze^x+ae^z\right)\end{pmatrix}\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \\
&= \left(-2e^y+ce^x+5e^z-be^y+3e^x+ae^z\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z
\end{aligned}
$$

$$\begin{cases}(c+3)\,e^x = 0\\ (-2-b)\,e^y = 0\\ (5+a)\,e^z = 0\end{cases} \implies \begin{cases}a = -5\\ b = -2\\ c = -3\end{cases}$$

10.3

(1)

$$\begin{aligned}
\mathrm{d}\alpha &= \mathrm{d}\left(-\frac{y}{\sqrt{x^2+y^2}}\mathrm{d}x + \frac{x}{\sqrt{x^2+y^2}}\mathrm{d}y\right)\\
&= \mathrm{d}\left(-\frac{y}{\sqrt{x^2+y^2}}\right)\wedge \mathrm{d}x + \mathrm{d}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\wedge \mathrm{d}y\\
&= \left(\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}\mathrm{d}x - \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}\mathrm{d}y\right)\wedge \mathrm{d}x + \left(\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}\mathrm{d}x - \frac{xy}{(x^2+y^2)^{\frac{3}{2}}}\mathrm{d}y\right)\wedge \mathrm{d}y\\
&= \left(\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} + \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}\right)\mathrm{d}x\wedge \mathrm{d}y\\
&= \frac{1}{\sqrt{x^2+y^2}}\mathrm{d}x\wedge \mathrm{d}y
\end{aligned}$$

(2)

$\alpha = \mathrm{d}f$ をみたす $0-$ 形式 f が存在すると、$\mathrm{d}\alpha = \mathrm{d}\,(\mathrm{d}f) = 0$
$\dfrac{1}{\sqrt{x^2+y^2}} \neq 0$ だから、$0-$ 形式 f は存在しない

10.4

(1)

$$\begin{aligned}
\mathrm{d}\alpha &= \mathrm{d}x_1\wedge \mathrm{d}x_3 + \mathrm{d}x_2\wedge \mathrm{d}x_4 + \mathrm{d}x_1\wedge \mathrm{d}x_3 + \mathrm{d}x_2\wedge \mathrm{d}x_4\\
&= 2\mathrm{d}x_1\wedge \mathrm{d}x_3 + 2\mathrm{d}x_2\wedge \mathrm{d}x_4
\end{aligned}$$

(2)

$\mathrm{d}\alpha = \mathrm{d}f\wedge \mathrm{d}g$ で、$\mathrm{d}\alpha = 2\mathrm{d}x_1\wedge \mathrm{d}x_3 + 2\mathrm{d}x_2\wedge \mathrm{d}x_4$ だから
条件をみたす $0-$ 形式 f, g は存在しない

10.1

(1)

$$\begin{aligned}
\mathrm{d}\left(e^x\cos y\mathrm{d}x - e^x\sin y\mathrm{d}y\right) &= \left(e^x\cos y\mathrm{d}x - e^x\sin y\mathrm{d}y\right)\wedge \mathrm{d}x - \left(e^x\sin y\mathrm{d}x + e^x\cos y\mathrm{d}y\right)\wedge \mathrm{d}y\\
&= e^x\sin y\mathrm{d}x\wedge \mathrm{d}y - e^x\sin y\mathrm{d}x\wedge \mathrm{d}y = 0
\end{aligned}$$

(2)

$$
\begin{aligned}
\mathrm{d}\left(13x\mathrm{d}x + y^2\mathrm{d}y + xyz\mathrm{d}z\right) &= \left(\nabla \times \begin{pmatrix} 13x \\ y^2 \\ xyz \end{pmatrix}\right) \cdot \begin{pmatrix} \mathrm{d}y \wedge \mathrm{d}z \\ \mathrm{d}z \wedge \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix} \\
&= \begin{pmatrix} xz \\ -yz \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}y \wedge \mathrm{d}z \\ \mathrm{d}z \wedge \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix} \\
&= xz\mathrm{d}y \wedge \mathrm{d}z - yz\mathrm{d}z \wedge \mathrm{d}x
\end{aligned}
$$

(3)

$$
\begin{aligned}
\mathrm{d}\left(\mathrm{d}x \wedge \mathrm{d}y\right) &= \mathrm{d}\left(\mathrm{d}x\right) \wedge \mathrm{d}y - \mathrm{d}x \wedge \mathrm{d}\left(\mathrm{d}y\right) \\
&= 0 \wedge \mathrm{d}y - \mathrm{d}x \wedge 0 \\
&= 0
\end{aligned}
$$

(4)

$$
\begin{aligned}
\mathrm{d}\left(z^2\mathrm{d}x \wedge \mathrm{d}y + \left(z^2 + 2y\right)\mathrm{d}x \wedge \mathrm{d}z\right) &= \mathrm{d}\left(\begin{pmatrix} 0 \\ -\left(z^2 + 2y\right) \\ z^2 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}y \wedge \mathrm{d}z \\ \mathrm{d}z \wedge \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix}\right) \\
&= \left(\nabla \cdot \begin{pmatrix} 0 \\ -\left(z^2 + 2y\right) \\ z^2 \end{pmatrix}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= (2z - 2)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z
\end{aligned}
$$

(5)

$$
\begin{aligned}
\mathrm{d}\left(z\mathrm{d}x \wedge \mathrm{d}y + y\mathrm{d}x \wedge \mathrm{d}z + x\mathrm{d}y \wedge \mathrm{d}z\right) &= \left(\nabla \cdot \begin{pmatrix} x \\ -y \\ z \end{pmatrix}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z
\end{aligned}
$$

10.2

(1)

$$
\begin{aligned}
\mathrm{d}\alpha &= -\mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}x_3 \wedge \mathrm{d}x_4 - \mathrm{d}x_3 \wedge \mathrm{d}x_4 \\
&= -2\mathrm{d}x_1 \wedge \mathrm{d}x_2 - 2\mathrm{d}x_3 \wedge \mathrm{d}x_4
\end{aligned}
$$

(2)

$\mathrm{d}\alpha = \theta \wedge \phi$ をみたす θ, ϕ が存在すると仮定すると、$\mathrm{d}\alpha \wedge \mathrm{d}\alpha = 0$ より

$$
\begin{aligned}
0 &= (-2\mathrm{d}x_1 \wedge \mathrm{d}x_2 - 2\mathrm{d}x_3 \wedge \mathrm{d}x_4) \wedge (-2\mathrm{d}x_1 \wedge \mathrm{d}x_2 - 2\mathrm{d}x_3 \wedge \mathrm{d}x_4) \\
&= 4\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 + 4\mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \\
&= 8\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \\
&\neq 0
\end{aligned}
$$

矛盾するから、条件を満たす θ, ϕ は存在しない