11.1

$$\omega = \left(x^2+y^2\right)\mathrm{d}x\wedge\mathrm{d}y + \mathrm{d}x\wedge\mathrm{d}z - \mathrm{d}y\wedge\mathrm{d}z$$
$$\phi = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$$

(1)

$$\phi^*\omega = \left(u^2+v^2\right)\mathrm{d}u\wedge\mathrm{d}v + \mathrm{d}u\wedge\mathrm{d}0 - \mathrm{d}v\wedge\mathrm{d}0 \tag{1}$$
$$= \left(u^2+v^2\right)\mathrm{d}u\wedge\mathrm{d}v \tag{2}$$

(2)

$$\int_{\phi|K}\omega = \iint_{\{u^2+v^2<1\}}\mathrm{d}u\mathrm{d}v \tag{3}$$
$$= \int_0^1\int_0^{2\pi} r^2\cdot r\mathrm{d}\theta\mathrm{d}\rho \tag{4}$$
$$= 2\pi\int_0^1 r^3\mathrm{d}r \tag{5}$$
$$= \frac{\pi}{2} \tag{6}$$

11.2

$$\omega = \frac{z}{\sqrt{x^2+y^2+z^2}}\mathrm{d}x\wedge\mathrm{d}y - \frac{y}{\sqrt{x^2+y^2+z^2}}\mathrm{d}x\wedge\mathrm{d}z + \frac{x}{\sqrt{x^2+y^2+z^2}}\mathrm{d}y\wedge\mathrm{d}z$$
$$\phi(u,v) = \begin{pmatrix} \sin u\cos v \\ \sin u\sin v \\ \cos u \end{pmatrix}$$

(1)

$$\phi^*\omega = z\mathrm{d}x\wedge\mathrm{d}y + y\mathrm{d}z\wedge\mathrm{d}x + x\mathrm{d}y\wedge\mathrm{d}z \tag{7}$$

ここで $\begin{cases} \mathrm{d}x = \cos u\cos v\mathrm{d}u - \sin u\sin v\mathrm{d}v \\ \mathrm{d}y = \cos u\sin v\mathrm{d}u + \sin u\cos v\mathrm{d}v \\ \mathrm{d}z = -\sin u\mathrm{d}u \end{cases}$

$$\begin{aligned} \mathrm{d}x\wedge\mathrm{d}y &= (\cos u\cos v\mathrm{d}u - \sin u\sin v\mathrm{d}v)\wedge(\cos u\sin v\mathrm{d}u + \sin u\cos v\mathrm{d}v) \\ &= \sin u\cos u\cos^2 v\mathrm{d}u\wedge\mathrm{d}v - \sin u\cos u\sin^2 v\mathrm{d}v\wedge\mathrm{d}u \\ &= \sin u\cos u\mathrm{d}u\wedge\mathrm{d}v \\ \mathrm{d}y\wedge\mathrm{d}z &= (\cos u\sin v\mathrm{d}u + \sin u\cos v\mathrm{d}v)\wedge(-\sin u\mathrm{d}u) \\ &= \sin^2 u\cos v\mathrm{d}u\wedge\mathrm{d}v \end{aligned}$$

$$\begin{aligned} \mathrm{d}z\wedge\mathrm{d}x &= (-\sin u\mathrm{d}u)\wedge(\cos u\cos v\mathrm{d}u - \sin u\sin v\mathrm{d}v) \\ &= \sin^2 u\sin v\mathrm{d}u\wedge\mathrm{d}v \end{aligned}$$

$$
\begin{aligned}
\phi^*\omega &= z\mathrm{d}x \wedge \mathrm{d}y + y\mathrm{d}z \wedge \mathrm{d}x + x\mathrm{d}y \wedge \mathrm{d}z & (8)\\
&= \sin u \cos^2 u\mathrm{d}u \wedge \mathrm{d}v + \sin^3 u \sin^2 v\mathrm{d}u \wedge \mathrm{d}v + \sin^3 u \cos^2 v\mathrm{d}u \wedge \mathrm{d}v & (9)\\
&= \sin u \left(\cos^2 u + \sin^2 u \sin^2 v + \sin^2 u \cos^2 v\right) \mathrm{d}u \wedge \mathrm{d}v & (10)\\
&= \sin u\mathrm{d}u \wedge \mathrm{d}v & (11)
\end{aligned}
$$

(2)

$$
\begin{aligned}
\int_{\phi|K} \omega &= \iint_{(0,\pi)\times(0,2\pi)} \sin u\mathrm{d}u\mathrm{d}v & (12)\\
&= \int_0^{2\pi}\int_0^{\pi} \sin u\mathrm{d}u\mathrm{d}v & (13)\\
&= 4\pi & (14)
\end{aligned}
$$

11.1

(1)

$\mathrm{d}z = 2u\mathrm{d}u - 2v\mathrm{d}v$ だから、$\begin{cases}\mathrm{d}z \wedge \mathrm{d}x = (2u\mathrm{d}u - 2v\mathrm{d}v) \wedge \mathrm{d}u = 2v\mathrm{d}u \wedge \mathrm{d}v \\ \mathrm{d}y \wedge \mathrm{d}z = \mathrm{d}v \wedge (2u\mathrm{d}u - 2v\mathrm{d}v) = -2u\mathrm{d}u \wedge \mathrm{d}v\end{cases}$

$$
\begin{aligned}
\phi^*\omega &= -\left(v^2 + uv\right)\mathrm{d}u \wedge \mathrm{d}v - 2v\left(u - u^2v + v^3\right)\mathrm{d}u \wedge \mathrm{d}v - 2u\left(v + u^3 - uv^2\right)\mathrm{d}u \wedge \mathrm{d}v & (15)\\
&= \left(-v^2 - uv - 2uv + 2u^2v^2 - 2v^4 - 2uv - 2u^4 + 2u^2v^2\right)\mathrm{d}u \wedge \mathrm{d}v & (16)\\
&= \left(-2u^4 - 5uv - v^2 + 4u^2v^2 - 2v^4\right)\mathrm{d}u \wedge \mathrm{d}v & (17)
\end{aligned}
$$

(2)

$$
\begin{aligned}
\int_{\phi|K} \omega &= \iint_{\{u^2+v^2<9\}} \left(-2u^4 - 5uv - v^2 + 4u^2v^2 - 2v^4\right)\mathrm{d}u\mathrm{d}v & (18)\\
&= \int_{-3}^{3}\int_{-\sqrt{9-u^2}}^{\sqrt{9-u^2}} \left(-2u^4 - 5uv - v^2 + 4u^2v^2 - 2v^4\right)\mathrm{d}v\mathrm{d}u & (19)\\
&= -\frac{2}{15}\int_{-3}^{3}\sqrt{9-u^2}\left(531 - 293u^2 + 56u^4\right)\mathrm{d}u & (20)\\
&= -\frac{1053}{4}\pi & (21)
\end{aligned}
$$

11.2

(1)

ここで $\begin{cases}\mathrm{d}x = \cos u \cos v\mathrm{d}u - \sin u \sin v\mathrm{d}v \\ \mathrm{d}y = \cos u \sin v\mathrm{d}u + \sin u \cos v\mathrm{d}v \\ \mathrm{d}z = -\sin u\mathrm{d}u\end{cases}$

$$
\begin{aligned}
\mathrm{d}x \wedge \mathrm{d}y &= (\cos u \cos v\mathrm{d}u - \sin u \sin v\mathrm{d}v) \wedge (\cos u \sin v\mathrm{d}u + \sin u \cos v\mathrm{d}v)\\
&= \sin u \cos u \cos^2 v\mathrm{d}u \wedge \mathrm{d}v - \sin u \cos u \sin^2 v\mathrm{d}v \wedge \mathrm{d}u\\
&= \sin u \cos u\mathrm{d}u \wedge \mathrm{d}v\\
\mathrm{d}y \wedge \mathrm{d}z &= (\cos u \sin v\mathrm{d}u + \sin u \cos v\mathrm{d}v) \wedge (-\sin u\mathrm{d}u)\\
&= \sin^2 u \cos v\mathrm{d}u \wedge \mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}z \wedge \mathrm{d}x &= (-\sin u \mathrm{d}u) \wedge (\cos u \cos v \mathrm{d}u - \sin u \sin v \mathrm{d}v) \\
&= \sin^2 u \sin v \mathrm{d}u \wedge \mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
\phi^* \omega &= z^3 \mathrm{d}x \wedge \mathrm{d}y + y^3 \mathrm{d}z \wedge \mathrm{d}x + x^3 \mathrm{d}y \wedge \mathrm{d}z && (22) \\
&= \left(\sin u \cos^4 u + \sin^5 u \sin^4 v + \sin^5 u \cos^4 v\right) \mathrm{d}u \wedge \mathrm{d}v && (23) \\
&= \sin u \left(\frac{1}{4} \sin^4 u \left(\cos 4v + 3\right) + \cos^4 u\right) \mathrm{d}u \wedge \mathrm{d}v && (24)
\end{aligned}
$$

(2)

$$
\begin{aligned}
\int_{\phi|K} \omega &= \iint_{(0,\pi)\times(0,2\pi)} \sin u \left(\frac{1}{4} \sin^4 u \left(\cos 4v + 3\right) + \cos^4 u\right) \mathrm{d}u \mathrm{d}v && (25) \\
&= \int_0^{\pi} \frac{1}{4} \sin^5 u \int_0^{2\pi} \left(\cos 4v + 3\right) \mathrm{d}v \mathrm{d}u + \int_0^{\pi} \int_0^{2\pi} \sin u \cos^4 u \mathrm{d}v \mathrm{d}u && (26) \\
&= \frac{3}{2}\pi \int_0^{\pi} \sin^5 u \mathrm{d}u + 2\pi \int_0^{\pi} \sin u \cos^4 u \mathrm{d}u && (27) \\
&= \frac{8}{5}\pi + \frac{4}{5}\pi && (28) \\
&= \frac{12}{5}\pi && (29)
\end{aligned}
$$