12.1

$$\begin{aligned}\alpha(\gamma(t)) &= \frac{rt\cos t}{r}\mathrm{d}(r\cos t)+\frac{rt\sin t}{r}\mathrm{d}(r\sin t)+r\mathrm{d}t\\ &= t\cos t(-r\sin t\mathrm{d}t)+t\sin t(r\cos t\mathrm{d}t)+r\mathrm{d}t\\ &= -rt\sin t\cos t\mathrm{d}t+rt\sin t\cos t\mathrm{d}t+r\mathrm{d}t\\ &= r\mathrm{d}t\end{aligned}$$

$$\begin{aligned}\int_\gamma \alpha &= \int_0^{2\pi} r\mathrm{d}t\\ &= 2\pi r\end{aligned}$$

12.2

$$\begin{aligned}\phi^*\omega &= -u^2v\mathrm{d}u+u^3\mathrm{d}v+u^2\left(u^2+v^2\right)\mathrm{d}\left(u^2+v^2\right)\\ &= -u^2v\mathrm{d}u+u^3\mathrm{d}v+u^2\left(u^2+v^2\right)(2u\mathrm{d}u+2v\mathrm{d}v)\\ &= -u^2v\mathrm{d}u+u^3\mathrm{d}v+2u^3\left(u^2+v^2\right)\mathrm{d}u+2u^2v\left(u^2+v^2\right)\mathrm{d}v\\ &= \left(2u^5+2u^3v^2-u^2v\right)\mathrm{d}u+\left(2u^4v+2u^2v^3+u^3\right)\mathrm{d}v\end{aligned}$$

$$\int_{\phi|\overline{\Omega}}\mathrm{d}\omega = \int_{\{u^2+v^2=4\}}\left(\left(2u^5+2u^3v^2-u^2v\right)\mathrm{d}u+\left(2u^4v+2u^2v^3+u^3\right)\mathrm{d}v\right)$$

$$\begin{cases}\left(2u^5+2u^3v^2-u^2v\right)\mathrm{d}u = \left(64\cos^5 t+64\sin^2 t\cos^3 t-8\sin t\cos^2 t\right)(-2\sin t\mathrm{d}t)\\ \left(2u^4v+2u^2v^3+u^3\right)\mathrm{d}v = \left(64\sin t\cos^4 t+64\sin^3 t\cos^2 t+8\cos^3 t\right)(2\cos t\mathrm{d}t)\end{cases}$$

$$\begin{cases}\left(2u^5+2u^3v^2-u^2v\right)\mathrm{d}u = -16\left(8\sin t\cos^5 t+8\sin^3 t\cos^3 t-\sin^2 t\cos^2 t\right)\mathrm{d}t\\ \left(2u^4v+2u^2v^3+u^3\right)\mathrm{d}v = 16\left(8\sin t\cos^5 t+8\sin^3 t\cos^3 t+\cos^4 t\right)\mathrm{d}t\end{cases}$$

$$\Longrightarrow \left(2u^5+2u^3v^2-u^2v\right)\mathrm{d}u+\left(2u^4v+2u^2v^3+u^3\right)\mathrm{d}v = 16\sin^2 t\cos^2 t+16\cos^4 t$$

$$\begin{aligned}\int_{\phi|\overline{\Omega}}\mathrm{d}\omega &= \int_0^{2\pi}\left(16\sin^2 t\cos^2 t+16\cos^4 t\right)\mathrm{d}t\\ &= \int_0^{2\pi}\cos^2 t\mathrm{d}t\\ &= \pi\end{aligned}$$

12.3

$$\int_S(\alpha+\mathrm{d}f)\wedge(\beta+\mathrm{d}g) = \int_S(\alpha\wedge\beta+\mathrm{d}f\wedge\beta+\alpha\wedge\mathrm{d}g+\mathrm{d}f\wedge\mathrm{d}g)$$

$$\begin{cases} \mathrm{d}(\alpha g) = \mathrm{d}\alpha \wedge g - \alpha \wedge \mathrm{d}g = -\alpha \wedge \mathrm{d}g \\ \mathrm{d}(\beta f) = \mathrm{d}\beta \wedge f - \beta \wedge \mathrm{d}f = -\beta \wedge \mathrm{d}f \\ \mathrm{d}(f\mathrm{d}g) = \mathrm{d}f \wedge \mathrm{d}g - f \wedge \mathrm{d}(\mathrm{d}g) = \mathrm{d}f \wedge \mathrm{d}g \end{cases}$$

$$\begin{aligned} \int_S (\alpha + \mathrm{d}f) \wedge (\beta + \mathrm{d}g) &= \int_S (\alpha \wedge \beta + \mathrm{d}(\beta f) - \mathrm{d}(\alpha g) + \mathrm{d}(f\mathrm{d}g)) \\ &= \int_S (\alpha \wedge \beta + (\nabla \times (\beta f)) - (\nabla \times (\alpha g)) + (\nabla \times (f\mathrm{d}g))) \\ &= \int_S \alpha \wedge \beta \end{aligned}$$

12.4

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}\left(\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}y \wedge \mathrm{d}z \\ \mathrm{d}z \wedge \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix} \right) \\ &= \frac{2}{\sqrt{x^2+y^2+z^2}} \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \end{aligned}$$

$$\begin{aligned} \int_{\partial V} \omega &= \int_{V} \mathrm{d}\omega \\ &= \int_0^{2r}\int_0^{\pi}\int_0^{2\pi} \frac{2}{\rho} \cdot \rho^2 \sin\phi \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}\rho - \int_0^{r}\int_0^{\pi}\int_0^{2\pi} \frac{2}{\rho} \cdot \rho^2 \sin\phi \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}\rho \\ &= 2\pi \left(\int_0^{2r}\int_0^{\pi} 2\rho \sin\phi \mathrm{d}\phi \mathrm{d}\rho - \int_0^{r}\int_0^{\pi} 2\rho \sin\phi \mathrm{d}\phi \mathrm{d}\rho \right) \\ &= 8\pi \left(\int_0^{2r} \rho \mathrm{d}\rho - \int_0^{r} \rho \mathrm{d}\rho \right) \\ &= 12\pi r^2 \end{aligned}$$

12.1

$\alpha = \mathrm{d}f$ とすると、$\begin{cases} \frac{\partial f}{\partial x} = x\left(r^2 - z^2\right) \\ \frac{\partial f}{\partial y} = y\left(r^2 - z^2\right) \\ \frac{\partial f}{\partial z} = -z\left(x^2 + y^2\right) \end{cases} \quad \Longrightarrow f = \int x\left(r^2 - z^2\right)\mathrm{d}x = \frac{1}{2}x^2\left(r^2 - z^2\right) + g(y,z)$

これを $\frac{\partial f}{\partial y} = y\left(r^2 - z^2\right)$ に代入すると $\frac{\partial}{\partial y}\left[\frac{1}{2}x^2\left(r^2 - z^2\right) + g(y,z)\right] = y\left(r^2 - z^2\right)$

$\Longleftrightarrow \frac{\partial}{\partial y} g(y,z) = y\left(r^2 - z^2\right) \Longleftrightarrow g(y,z) = \int y\left(r^2 - z^2\right)\mathrm{d}y = \frac{1}{2}y^2\left(r^2 - z^2\right) + h(z)$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[\frac{1}{2}x^2\left(r^2 - z^2\right) + \frac{1}{2}y^2\left(r^2 - z^2\right) + h(z)\right] = -\left(x^2 + y^2\right)z + \frac{\partial}{\partial z}h(z) = -z\left(x^2 + y^2\right)$

$\Longrightarrow h(z) = 0$

以上より、$\alpha = \mathrm{d}f$ をみたす $f = \dfrac{1}{2}\left(x^2+y^2\right)\left(r^2-z^2\right)$

$$\begin{aligned}\int_\gamma \alpha &= f\left(\gamma\left(2\pi\right)\right) - f\left(\gamma\left(0\right)\right)\\ &= f\left(R+r,0,0\right) - f\left(R+r,0,0\right) = 0\end{aligned}$$

12.2

$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}\left(\begin{pmatrix} x\sin\frac{z}{k} - y\cos\frac{z}{k} \\ x\cos\frac{z}{k} + y\sin\frac{z}{k} \\ x^2+y^2+\frac{z^2}{k^2}\end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\\ \mathrm{d}z\end{pmatrix}\right)\\ &= \begin{pmatrix} 2y + \dfrac{x\sin\frac{z}{k}}{k} - \dfrac{y\cos\frac{z}{k}}{k} \\ -2x + \dfrac{x\cos\frac{z}{k}}{k} + \dfrac{y\sin\frac{z}{k}}{k} \\ 2\cos\dfrac{z}{k}\end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}y\wedge\mathrm{d}z\\ \mathrm{d}z\wedge\mathrm{d}x\\ \mathrm{d}x\wedge\mathrm{d}y\end{pmatrix}\end{aligned}$$