4.1

$$\sigma = \begin{pmatrix} u \\ v \\ \sqrt{r^2-u^2-v^2} \end{pmatrix}, \sigma_u = \begin{pmatrix} 1 \\ 0 \\ -\frac{u}{\sqrt{r^2-u^2-v^2}} \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 1 \\ -\frac{v}{\sqrt{r^2-u^2-v^2}} \end{pmatrix}$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} \frac{u}{\sqrt{r^2-u^2-v^2}} \\ \frac{v}{\sqrt{r^2-u^2-v^2}} \\ 1 \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \frac{r}{\sqrt{r^2-u^2-v^2}}$$

(1)

$$\begin{aligned} Area\,(T) &= \iint_\Omega \|\sigma_u \times \sigma_v\| \,\mathrm{d}u\mathrm{d}v \\ &= \iint_{u^2+v^2<a^2} \frac{r}{\sqrt{r^2-u^2-v^2}}\mathrm{d}u\mathrm{d}v \\ &= \int_0^{2\pi}\int_0^a \frac{r}{\sqrt{r^2-\delta^2}} \cdot \delta\mathrm{d}\delta\mathrm{d}\theta \\ &= \int_0^{2\pi} \left(r^2 - r\sqrt{r^2-\sqrt{a}}\right)\mathrm{d}\theta \\ &= 2\pi r\left(r - \sqrt{r^2-\sqrt{a}}\right) \end{aligned}$$

(2)

$$\begin{aligned} \iint_T f\mathrm{d}A &= \iint_\Omega \left(4u^2v^2\sqrt{r^2-u^2-v^2}\right) \cdot \frac{r}{\sqrt{r^2-u^2-v^2}}\mathrm{d}u\mathrm{d}v \\ &= 4r\iint_{u^2+v^2<a^2} u^2v^2\mathrm{d}u\mathrm{d}v \\ &= 4r\int_0^{2\pi}\int_0^a \delta^4\sin^2\theta\cos^2\theta \cdot \delta\mathrm{d}\delta\mathrm{d}\theta \\ &= 4r\int_0^{2\pi}\sin^2\theta\cos^2\theta\left(\int_0^a \delta^5\mathrm{d}\delta\right)\mathrm{d}\theta \\ &= \frac{2}{3}ra^6\int_0^{2\pi}\sin^2\theta\cos^2\theta\mathrm{d}\theta \\ &= \frac{1}{6}ra^6\int_0^{2\pi}\sin^2 2\theta\mathrm{d}\theta \\ &= \frac{1}{12}ra^6\int_0^{2\pi}(1-\cos 4\theta)\,\mathrm{d}\theta \\ &= \frac{1}{6}ra^6\pi \end{aligned}$$

(3)

$$
\begin{aligned}
\iint_T \mathbf{v}_1 \cdot \mathrm{d}\mathbf{A} &= \iint_\Omega \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} \dfrac{u}{\sqrt{r^2 - u^2 - v^2}} \\ \dfrac{v}{\sqrt{r^2 - u^2 - v^2}} \\ 1 \end{pmatrix} \mathrm{d}u\mathrm{d}v \\
&= 3 \int_0^{2\pi} \int_0^a \delta \mathrm{d}\delta \mathrm{d}\theta \\
&= 3 \int_0^{2\pi} \frac{1}{2} a^2 \mathrm{d}\theta \\
&= 3\pi a^2
\end{aligned}
$$

(4)

$$
\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a\cos t \\ a \sin t \end{pmatrix}, C(t) = \sigma(u, v) = \begin{pmatrix} a\cos t \\ a\sin t \\ \sqrt{r^2 + a^2} \end{pmatrix}
$$

$$
\mathbf{v}_2(C(t)) = \begin{pmatrix} a\cos t - 3a\sin t \\ a\sin t\left(r^2 - a^2\right) \\ a^2 \sin^2 t \sqrt{r^2 - a^2} \end{pmatrix}, C'(t) = \begin{pmatrix} -a\sin t \\ a\cos t \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
\iint_{\sigma(\overline{\Omega})} \nabla \times \mathbf{v}_2 \cdot \mathrm{d}\mathbf{A} &= \oint_{\partial T} \mathbf{v}_2 \cdot \mathrm{d}\mathbf{r} \\
&= \int_0^{2\pi} \left(a^2\left(r^2 - a^2 - a\right)\sin t \cos t + 3a^2 \sin^2 t\right) \mathrm{d}t \\
&= a^2\left(r^2 - a^2 - 1\right) \int_0^{2\pi} \sin t \cos t \mathrm{d}t + 3a^2 \int_0^{2\pi} \sin^2 t \mathrm{d}t \\
&= a^2\left(r^2 - a^2 - 1\right)\left[-\frac{1}{4}\cos 2t\right]_0^{2\pi} + 3a^2\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{2\pi} \\
&= 3\pi a^2
\end{aligned}
$$

(5)

$$\mathbf{v}_3\left(\sigma\left(u,v\right)\right)=\begin{pmatrix} u\sqrt{r^2-u^2-v^2} \\ v\sqrt{r^2-u^2-v^2} \\ r^2-\left(r^2-u^2-v^2\right) \end{pmatrix}=\begin{pmatrix} u\sqrt{r^2-u^2-v^2} \\ v\sqrt{r^2-u^2-v^2} \\ u^2+v^2 \end{pmatrix},\sigma_u\times\sigma_v=\begin{pmatrix} \frac{u}{\sqrt{r^2-u^2-v^2}} \\ \frac{v}{\sqrt{r^2-u^2-v^2}} \\ 1 \end{pmatrix}$$

$$\begin{aligned}\iint_T \mathbf{v}_3\cdot\mathrm{d}\mathbf{A} &= \iint_\Omega \begin{pmatrix} u\sqrt{r^2-u^2-v^2} \\ v\sqrt{r^2-u^2-v^2} \\ u^2+v^2 \end{pmatrix}\cdot\begin{pmatrix} \frac{u}{\sqrt{r^2-u^2-v^2}} \\ \frac{v}{\sqrt{r^2-u^2-v^2}} \\ 1 \end{pmatrix}\mathrm{d}u\mathrm{d}v \\ &= 2\iint_\Omega\left(u^2+v^2\right)\mathrm{d}u\mathrm{d}v \\ &= 2\int_0^{2\pi}\int_0^a \delta^2\cdot\delta\mathrm{d}\delta\mathrm{d}t \\ &= \frac{1}{2}a^4\int_0^{2\pi}\mathrm{d}t \\ &= a^4\pi\end{aligned}$$

(6)

$$C\left(t\right)=\sigma\left(u,v\right)=\begin{pmatrix} a\cos t \\ a\sin t \\ \sqrt{r^2-a^2} \end{pmatrix},C'\left(t\right)=\begin{pmatrix} -a\sin t \\ a\cos t \\ 0 \end{pmatrix}$$

$$\mathbf{v}_4\left(C\left(t\right)\right)=\begin{pmatrix} a\sin t\cdot\left(r^2-a^2\right) \\ r^2\cdot a\cos t \\ a^2\sin t\cos t\cdot\sqrt{r^2-a^2} \end{pmatrix}$$

$$\begin{aligned}\iint_{\sigma(\overline{\Omega})}\nabla\times\mathbf{v}_4\cdot\mathrm{d}\mathbf{A} &= \oint_{\partial T}\mathbf{v}_4\cdot\mathrm{d}r \\ &= \oint_{\partial T}\begin{pmatrix} a\sin t\cdot\left(r^2-a^2\right) \\ r^2\cdot a\cos t \\ a^2\sin t\cos t\cdot\sqrt{r^2-a^2} \end{pmatrix}\cdot\begin{pmatrix} -a\sin t \\ a\cos t \\ 0 \end{pmatrix}\mathrm{d}t \\ &= \int_0^{2\pi}\left(-a^2\left(r^2-a^2\right)\sin^2 t+a^2r^2\cos^2 t\right)\mathrm{d}t \\ &= \left(a^4-a^2r^2\right)\int_0^{2\pi}\sin^2 t\mathrm{d}t+a^2r^2\int_0^{2\pi}\cos^2 t\mathrm{d}t \\ &= \frac{1}{2}\left(a^4-a^2r^2\right)\left[t-\frac{1}{2}\sin 2t\right]_0^{2\pi}+\frac{1}{2}a^2r^2\left[t+\frac{1}{2}\sin 2t\right]_0^{2\pi} \\ &= \pi a^4\end{aligned}$$

4.1

$$\sigma_u=\begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix},\sigma_v=\begin{pmatrix} 0 \\ 1 \\ -2v \end{pmatrix},\sigma_u\times\sigma_v=\begin{pmatrix} -2u \\ 2v \\ 1 \end{pmatrix},\|\sigma_u\times\sigma_v\|=\sqrt{4u^2+4v^2+1}$$

(1)

$$
\begin{aligned}
Area\,(T) &= \iint_{\Omega} \|\sigma_u \times \sigma_v\| \, \mathrm{d}u\mathrm{d}v \\
&= \iint_{\Omega} \sqrt{4u^2 + 4v^2 + 1}\mathrm{d}u\mathrm{d}v \\
&= \int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}} \sqrt{4\delta^2 + 1} \cdot \delta \mathrm{d}\delta \mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^{\frac{3}{4}} \frac{1}{2}\sqrt{4t+1}\mathrm{d}t\mathrm{d}\theta \\
&= \frac{1}{12}\int_0^{2\pi} \left[(4t+1)^{\frac{3}{2}}\right]_0^{\frac{3}{4}} \mathrm{d}\theta \\
&= \frac{1}{12}\int_0^{2\pi} 7\mathrm{d}\theta \\
&= \frac{7}{6}\pi
\end{aligned}
$$

(2)

$$
\begin{aligned}
\iint_T f\mathrm{d}A &= \iint_{\Omega} \left(u^2 + v^2\right) \sqrt{4u^2 + 4v^2 + 1}\mathrm{d}u\mathrm{d}v \\
&= \int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}} \delta^2 \cdot \sqrt{4\delta^2 + 1} \cdot \delta \mathrm{d}\delta \mathrm{d}\theta \\
&= \frac{1}{32}\int_0^{2\pi} \int_1^4 \left(t^{\frac{3}{2}} - t^{\frac{1}{2}}\right) \mathrm{d}t\mathrm{d}\theta \\
&= \frac{1}{32}\int_0^{2\pi} \left[\frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}}\right]_1^4 \mathrm{d}\theta \\
&= \frac{1}{32}\int_0^{2\pi} \frac{116}{25}\mathrm{d}\theta \\
&= \frac{29}{60}\pi
\end{aligned}
$$

(3)

$$
\begin{aligned}
\iint_T \mathbf{v}_1 \cdot \mathrm{d}\mathbf{A} &= \iint_\Omega \begin{pmatrix} -u\left(u^2-v^2\right) \\ v\left(u^2-v^2\right) \\ u^2+v^2 \end{pmatrix} \cdot \begin{pmatrix} -2u \\ 2v \\ 1 \end{pmatrix} \mathrm{d}u\mathrm{d}v \\
&= \iint_\Omega \left(u^2+v^2\right)\left(2u^2-2v^2+1\right)\mathrm{d}u\mathrm{d}v \\
&= \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}} k^2\left(2k^2\cos 2\theta+1\right)\cdot k\mathrm{d}k\mathrm{d}\theta \\
&= \int_0^{2\pi}\left(\cos 2\theta\left[\frac{1}{3}k^6\right]_0^{\frac{\sqrt{3}}{2}} + \left[\frac{1}{4}k^4\right]_0^{\frac{\sqrt{3}}{2}}\right)\mathrm{d}\theta \\
&= \int_0^{2\pi}\left(\frac{9}{64}\cos 2\theta+\frac{9}{64}\right)\mathrm{d}\theta \\
&= \frac{9}{64}\int_0^{2\pi}(\cos 2\theta+1)\,\mathrm{d}\theta \\
&= \frac{9}{64}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{2\pi} \\
&= \frac{9}{32}\pi
\end{aligned}
$$

(4)

$$
C(t) = \sigma(u,v) = \begin{pmatrix} a\cos t \\ a\sin t \\ a^2\cos^2 t - a^2\sin^2 t \end{pmatrix} = \begin{pmatrix} a\cos t \\ a\sin t \\ a^2\cos 2t \end{pmatrix}, C'(t) = \begin{pmatrix} -a\sin t \\ a\cos t \\ -2a^2\sin 2t \end{pmatrix}
$$

$$
\begin{aligned}
\iint_{\sigma(\overline{\Omega})} \nabla\times\mathbf{v}_2\cdot\mathrm{d}\mathbf{A} &= \oint_{\partial T}\begin{pmatrix} -a\sin t \\ a\cos t \\ a^2\cos 2t \end{pmatrix}\cdot\begin{pmatrix} -a\sin t \\ a\cos t \\ -2a^2\sin 2t \end{pmatrix}\mathrm{d}t \\
&= a^2\int_0^{2\pi}\left(1-a^2\sin 4t\right)\mathrm{d}t \\
&= a^2\left[t+\frac{1}{4}a^2\cos 4t\right]_0^{2\pi} \\
&= 2a^2\pi
\end{aligned}
$$

4.2

$$
\sigma(u,v) = \begin{pmatrix} r\sin u\cos v \\ r\sin u\sin v \\ r\cos u \end{pmatrix}, \sigma_u = \begin{pmatrix} r\cos u\cos v \\ r\cos u\sin v \\ -r\sin u \end{pmatrix}, \sigma_v = \begin{pmatrix} -r\sin u\sin v \\ r\sin u\cos v \\ 0 \end{pmatrix}
$$

$$
\sigma_u\times\sigma_v = \begin{pmatrix} r^2\sin^2 u\cos v \\ r^2\sin^2 u\sin v \\ r^2\sin u\cos u \end{pmatrix}, \|\sigma_u\times\sigma_v\| = r^2\sin u
$$

(1)

$$
\begin{aligned}
Area\,(T) &= \iint_{\Omega} \|\sigma_u \times \sigma_v\| \,\mathrm{d}u\mathrm{d}v \\
&= \iint_{\Omega} r^2 \sin u \mathrm{d}u \mathrm{d}v \\
&= r^2 \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \sin u \mathrm{d}u \mathrm{d}v \\
&= r^2 \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \left[-\cos u\right]_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \mathrm{d}v \\
&= r^2 \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \mathrm{d}v \\
&= \pi r^2
\end{aligned}
$$

(2)

$$
\begin{aligned}
\iint_S f \mathrm{d}A &= \iint_D \left(r \sin u \cos v \cdot r \sin u \sin v \cdot r^2 \cos^2 u\right) \cdot \left(r^2 \sin u\right) \mathrm{d}u\mathrm{d}v \\
&= \iint_D \left(r^6 \sin^3 u \cos^2 u \sin v \cos v\right) \mathrm{d}u\mathrm{d}v \\
&= \int_0^{2\pi} \int_0^{\pi} r^6 \sin^3 u \cos^2 u \sin v \cos v \mathrm{d}u\mathrm{d}v \\
&= r^6 \int_0^{2\pi} \sin v \cos v \left(\int_0^{\pi} \sin^3 u \cos^2 u \mathrm{d}u \right) \mathrm{d}v \\
&= \frac{4}{15} r^6 \int_0^{2\pi} \sin v \cos v \mathrm{d}v \\
&= 0
\end{aligned}
$$

(3)

$$
\begin{aligned}
\iint_S \mathbf{v}_1 \cdot \mathrm{d}\mathbf{A} &= \iint_D \begin{pmatrix} 0 \\ 0 \\ r^3 \sin^3 u \cos^3 v \end{pmatrix} \cdot \begin{pmatrix} r^2 \sin^2 u \cos v \\ r^2 \sin^2 u \sin v \\ r^2 \sin u \cos u \end{pmatrix} \mathrm{d}u\mathrm{d}v \\
&= \iint_D r^5 \sin^4 u \cos u \cos^3 v \mathrm{d}u\mathrm{d}v \\
&= r^5 \int_0^{2\pi} \cos^3 v \left(\int_0^{\pi} \sin^4 u \cos u \mathrm{d}u \right) \mathrm{d}v \\
&= r^5 \int_0^{2\pi} 0 \cdot \cos^3 v \mathrm{d}v \\
&= 0
\end{aligned}
$$

(4)

$$\mathbf{v}_2 = \begin{pmatrix} yz^2 \\ xz^2 \\ 2xyz \end{pmatrix}, \nabla \times \mathbf{v}_2 = \begin{pmatrix} 2xz - 2xz \\ 2yz - 2yz \\ z^2 - z^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{から}$$

$$\iint_{\sigma(\overline{\Omega})} \nabla \times \mathbf{v}_2 \cdot \mathrm{d}\mathbf{A} = \oint_{\partial\Omega} 0 \cdot \mathrm{d}r = 0$$

4.3

$$\sigma(u, v) = \begin{pmatrix} u \\ v \\ u^2 + v^2 \end{pmatrix}, \sigma_u = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 1 \\ 2v \end{pmatrix}$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \sqrt{4u^2 + 4v^2 + 1}$$

(1)

$$\begin{aligned} Area\,(T) &= \iint_{\Omega} \|\sigma_u \times \sigma_v\| \,\mathrm{d}u\mathrm{d}v \\ &= \int_0^{2\pi} \int_2^9 \sqrt{4\delta^2 + 1}\delta \cdot \mathrm{d}\delta\mathrm{d}t \\ &= \frac{1}{2} \int_0^{2\pi} \left(1625\sqrt{13} - 17\sqrt{17}\right) \mathrm{d}t \\ &= \frac{1}{6} \left(1625\sqrt{13} - 17\sqrt{17}\right) \pi \end{aligned}$$

(2)

$$C_1(t) = \begin{pmatrix} 9\cos t \\ 9\sin t \\ 81 \end{pmatrix}, C_2(t) = \begin{pmatrix} 2\cos t \\ -2\sin t \\ 4 \end{pmatrix}, C_1'(t) = \begin{pmatrix} -9\sin t \\ 9\cos t \\ 0 \end{pmatrix}, C_2'(t) = \begin{pmatrix} -2\sin t \\ -2\cos t \\ 0 \end{pmatrix}$$

$$\mathbf{v}_1(C_1) = \begin{pmatrix} 81\cos^2 t + 9\sin t - 4 \\ 243\sin t\cos t \\ 1458\cos t + 6561 \end{pmatrix}, \mathbf{v}_1(C_2) = \begin{pmatrix} 8 - 4\cos^2 t \\ 8\sin t\cos t + 2\sin t \\ -4\sin t \end{pmatrix}$$

$$\begin{aligned} \iint_{\sigma(\overline{\Omega})} \nabla \times \mathbf{v}_1 \cdot \mathrm{d}\mathbf{A} &= \int_0^{2\pi} \left(1450\sin t\cos^2 t - 81\sin^2 t - 4\sin t\cos t + 20\sin t\right) \mathrm{d}t \\ &= -81\pi \end{aligned}$$

(3)

$$C_1\left(t\right)=\begin{pmatrix}9\cos t\\9\sin t\\81\end{pmatrix},C_2\left(t\right)=\begin{pmatrix}2\cos t\\-2\sin t\\4\end{pmatrix},C_1'\left(t\right)=\begin{pmatrix}-9\sin t\\9\cos t\\0\end{pmatrix},C_2'\left(t\right)=\begin{pmatrix}-2\sin t\\-2\cos t\\0\end{pmatrix}$$

$$\mathbf{v}_2\left(C_1\right)=\begin{pmatrix}162-81\cos^2 t\\-162\sin t\cos t-9\sin t\\18\sin t\end{pmatrix},\mathbf{v}_2\left(C_2\right)=\begin{pmatrix}8-4\cos^2 t\\8\sin t\cos t+2\sin t\\-4\sin t\end{pmatrix}$$

$$\begin{aligned}\iint_{\sigma(\overline{\Omega})}\nabla\times\mathbf{v}_2\cdot\mathrm{d}\mathbf{A}&=\int_0^{2\pi}\left(478\sin t\cos^2 t+14\sin t\cos t-340\sin t\right)\mathrm{d}t\\&=0\end{aligned}$$