7.1

(i)

$f(x,y,z) = \sqrt{x^2+y^2}, T_1 = \sigma((0,k)\times(0,2\pi))$

$(M1)$ $T = T_1$

$(M2)$ σ 局所パラメーター表示、$(0,k)\times(0,2\pi)$：面積確定、有界
$T_1 = \sigma((0,2k)\times(0,2\pi))$ より OK

$(M3)$ $i \neq j, i, j \in \{1\}$ は取れない. よって、OK

また $\sigma_u \times \sigma_v = \begin{pmatrix} k\sin v \\ k\cos v \\ u \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \sqrt{u^2+k^2}$

よって

$$\begin{aligned}\iint_T \mathrm{d}A &= \iint_{(0,k)\times(0,2\pi)} f(\sigma(u,v)) \|\sigma_u\times\sigma_v\| \mathrm{d}u\mathrm{d}v \\ &= \int_0^{2\pi}\int_0^k u\sqrt{u^2+k^2}\mathrm{d}u\mathrm{d}v \\ &= \frac{2}{3}\pi\left(2\sqrt{2}k^3 - k^3\right) \\ &= \frac{2}{3}\pi k^3\left(2\sqrt{2}-1\right)\end{aligned}$$

(ii)

$$\begin{aligned}Area(T) &= \iint_{(0,k)\times(0,2\pi)} \sqrt{u^2+k^2}\mathrm{d}u\mathrm{d}v \\ &= \int_0^{2\pi}\int_0^k \sqrt{u^2+k^2}\mathrm{d}u\mathrm{d}v \\ &= 2\pi k^2 \int_0^{\frac{\pi}{4}} \frac{1}{\cos^3\theta}\mathrm{d}\theta \\ &= \left(\sqrt{2}+\log\left(\sqrt{2}+1\right)\right)\pi k^2\end{aligned}$$

(iii)

$$\begin{aligned}\iint_T v\cdot \mathrm{d}A &= \iint_{(0,k)\times(0,2\pi)} v(\sigma(u,v))\cdot(\sigma_u\times\sigma_v)\mathrm{d}u\mathrm{d}v \\ &= \int_0^{2\pi}\int_0^k \left(-ku\cos 2v + k^2uv^2\right)\mathrm{d}u\mathrm{d}v \\ &= \frac{4}{3}\pi^3k^4\end{aligned}$$

7.2

(i)

$(M1)$ $S^2(1) = \sigma((0,\pi)\times(0,2\pi)) \cup \tau\left(\left\{\frac{\pi}{2}\right\}\times\left[\frac{\pi}{2},\frac{3}{2}\pi\right]\right)$,OK

$(M2)$ $(0,\pi)\times(0,2\pi)$ と $\left\{\frac{\pi}{2}\right\}\times\left[\frac{\pi}{2},\frac{3}{2}\pi\right]$ は面積確定な有界集合

$(M3)$ $\sigma^{-1}\left(\sigma\left((0,\pi)\times(0,2\pi)\right)\cap\tau\left(\left\{\frac{\pi}{2}\right\}\times\left[\frac{\pi}{2},\frac{3}{2}\pi\right]\right)\right)=\sigma^{-1}(\emptyset)$ は面積 0

$\sigma_u\times\sigma_v=\begin{pmatrix}\sin^2 u\cos v\\ \sin^2 u\sin v\\ \cos u\sin u\end{pmatrix}=\sin u\sigma(u,v)\,,\|\sigma(u,v)\|=\sin u$

$\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}=\sigma(u,v)=n(\sigma(u,v))$

同様に $\frac{\tau_u\times\tau_v}{\|\tau_u\times\tau_v\|}=\tau(u,v)=n(\tau(u,v))$

よって、σ,τ も正の向き

$$
\begin{aligned}
\iint_{S^2(1)} v\cdot\mathrm{d}A &= \iint_{(0,\pi)\times(0,2\pi)} v(\sigma(u,v))\cdot(\sigma_u\times\sigma_v)\,\mathrm{d}u\mathrm{d}v+\iint_{\left\{\frac{\pi}{2}\right\}\times\left[\frac{\pi}{2},\frac{3}{2}\pi\right]} v(\tau(u,v))\cdot(\tau_u\times\tau_v)\,\mathrm{d}u\mathrm{d}v\\
&=4\pi
\end{aligned}
$$

(ii)

$\nabla\cdot\omega=v$ なら、$\iint_{S^2(1)}(\nabla\cdot\omega)\,\mathrm{d}A=0$

(i) の結論と反する、このような ω は存在しない

7.1

(i)

$\sigma(u,v)=\begin{pmatrix}a\cosh u\cos v\\ a\cosh u\sin v\\ au\end{pmatrix}$ から、$T=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in S\middle|-1<z<1\right\}$ より、$u\in\left(-\frac{1}{a},\frac{1}{a}\right)$

$$
\begin{aligned}
Area(T) &= \iint_{\left(-\frac{1}{a},\frac{1}{a}\right)\times[0,2\pi]}\|\sigma_u\times\sigma_v\|\,\mathrm{d}u\mathrm{d}v\\
&=\int_{-\frac{1}{a}}^{\frac{1}{a}}\int_0^{2\pi}a^2\cosh^2 u\mathrm{d}v\mathrm{d}u\\
&=2a^2\pi\int_{-\frac{1}{a}}^{\frac{1}{a}}\cosh^2 u\mathrm{d}u\\
&=2a^2\pi\left(\frac{1}{a}+\frac{1}{2}\sinh\frac{2}{a}\right)\\
&=2a\pi+a^2\pi\sinh\frac{2}{a}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\iint_T v \cdot \mathrm{d}A &= \iint_{\left(-\frac{1}{a},\frac{1}{a}\right)\times[0,2\pi]} \begin{pmatrix} a\cosh u \sin v \\ -a\cosh u\cos v \\ a^2u^2 \end{pmatrix} \cdot \begin{pmatrix} -a^2\cosh u\cos v \\ -a^2\cosh u \sin v \\ a^2 \sinh u\cosh u \end{pmatrix} \mathrm{d}u\mathrm{d}v \\
&= \iint_{\left(-\frac{1}{a},\frac{1}{a}\right)\times[0,2\pi]} a^4u^2\sinh u\cosh u \mathrm{d}u\mathrm{d}v \\
&= a^4 \int_{-\frac{1}{a}}^{\frac{1}{a}} \int_0^{2\pi} u^2 \sinh u\cosh u \mathrm{d}v\mathrm{d}u \\
&= 2a^4\pi \int_{-\frac{1}{a}}^{\frac{1}{a}} u^2\sinh u\cosh u \mathrm{d}u \\
&= \frac{1}{4}a^4\pi\left[\left(2u^2+1\right)\cosh 2u - 2u\sinh 2u\right]_{-\frac{1}{a}}^{\frac{1}{a}} \\
&= 0
\end{aligned}
$$

7.2

(i)

$T_{R,r} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| \left(\sqrt{x^2+y^2}-R\right)^2 + z^2 = r^2 \right\}$

$z^2 = r^2 - \left(\sqrt{x^2+y^2}-R\right)^2 \geq 0$ から $R - r \leq \sqrt{x^2+y^2} \leq R + r$

よって、$\|\mathbf{x}\| = \sqrt{x^2+y^2+z^2} \leq \sqrt{(R+r)^2+z^2} \leq \sqrt{(R+r)^2+r^2}$

言い換えれば、$T_{R,r}$ は有界集合である.

また、$f(x,y,z) = \left(\sqrt{x^2+y^2}-R\right)^2 + z^2 - r^2$ とすると、$T_{R,r}$ は $f(x,y,z) = 0$ の開集合 $f^{-1}(\{0\})$ であるから、閉集合である

(ii)

$$
\sigma = \begin{pmatrix} (R+r\cos u)\cos v \\ (R+r\cos u)\sin v \\ r\sin u \end{pmatrix}, \sigma_u = \begin{pmatrix} -r\sin u\cos v \\ -r\sin u \sin v \\ r\cos u \end{pmatrix}, \sigma_v = \begin{pmatrix} -(R+r\cos u)\sin v \\ (R+r\cos u)\cos v \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
\sigma_u \times \sigma_v &= \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix} \times \begin{pmatrix} -(R+r\cos u)\sin v \\ (R+r\cos u)\cos v \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -r(R+r\cos u)\cos u\cos v \\ -r(R+r\cos u)\cos u\sin v \\ -r(R+r\cos u)\sin u \end{pmatrix}
\end{aligned}
$$

$\|\sigma_u \times \sigma_v\| = r\left(R + r\cos u\right)$

$$\begin{aligned}
Area\left(T\right) &= \iint_{[0,2\pi]\times[0,2\pi]} \|\sigma_u \times \sigma_v\| \,\mathrm{d}u\mathrm{d}v \\
&= \int_0^{2\pi}\int_0^{2\pi} r\left(R + r\cos u\right)\mathrm{d}v\mathrm{d}u \\
&= 2\pi \int_0^{2\pi} r\left(R + r\cos u\right)\mathrm{d}u \\
&= 4\pi^2 rR
\end{aligned}$$

(iii)

$\sigma\left(u,v\right) = \begin{pmatrix} \left(R + r\cos u\right)\cos v \\ \left(R + r\cos u\right)\sin v \\ r\sin u \end{pmatrix}$

$\sigma_u = \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix}, \sigma_v = \begin{pmatrix} -\left(R + r\cos u\right)\sin v \\ \left(R + r\cos u\right)\cos v \\ 0 \end{pmatrix}$

$$\begin{aligned}
\sigma_u \times \sigma_v &= \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix} \times \begin{pmatrix} -\left(R + r\cos u\right)\sin v \\ \left(R + r\cos u\right)\cos v \\ 0 \end{pmatrix} && (1) \\
&= \begin{pmatrix} -r\left(R + r\cos u\right)\cos u\cos v \\ -r\left(R + r\cos u\right)\cos u\sin v \\ -r\left(R + r\cos u\right)\sin u\cos^2 v - r\left(R + r\cos u\right)\sin u\sin^2 v \end{pmatrix} && (2) \\
&= \begin{pmatrix} -r\left(R + r\cos u\right)\cos u\cos v \\ -r\left(R + r\cos u\right)\cos u\sin v \\ -r\left(R + r\cos u\right)\sin u \end{pmatrix} && (3)
\end{aligned}$$

また

$$\begin{aligned}
v\left(\sigma\left(u,v\right)\right) &= v\left(\left(R + r\cos u\right)\cos v, \left(R + r\cos u\right)\sin v, r\sin u\right) && (4) \\
&= \begin{pmatrix} -\frac{\left(R + r\cos u\right)\sin v}{\left(R + r\cos u\right)^2} \\ \frac{\left(R + r\cos u\right)\cos v}{\left(R + r\cos u\right)^2} \\ \frac{r\sin u}{\left(R + r\cos u\right)^2} \end{pmatrix} && (5) \\
&= \begin{pmatrix} -\frac{\sin v}{R + r\cos u} \\ \frac{\cos v}{R + r\cos u} \\ \frac{r\sin u}{\left(R + r\cos u\right)^2} \end{pmatrix} && (6)
\end{aligned}$$

以上

$$\iint_S v \cdot \mathrm{d}A = \iint_{[0,2\pi]^2} \begin{pmatrix} -\dfrac{\sin v}{R + r\cos u} \\ \dfrac{\cos v}{R + r\cos u} \\ \dfrac{r\sin u}{(R + r\cos u)^2} \end{pmatrix} \cdot \begin{pmatrix} -r\,(R + r\cos u)\cos u\cos v \\ -r\,(R + r\cos u)\cos u\sin v \\ -r\,(R + r\cos u)\sin u \end{pmatrix} \mathrm{d}u\mathrm{d}v$$
$$= \int_0^{2\pi}\int_0^{2\pi}\left(r\cos u\sin v\cos v - r\cos u\sin v\cos v - \frac{r^2\sin^2 u}{R + r\cos u}\right)\mathrm{d}u\mathrm{d}v$$
$$= -r^2\int_0^{2\pi}\int_0^{2\pi}\frac{\sin^2 u}{R + r\cos u}\mathrm{d}u\mathrm{d}v$$
$$= -2\pi r^2\int_0^{2\pi}\frac{\sin^2 u}{R + r\cos u}\mathrm{d}u$$

ここで、$\displaystyle\int_0^{2\pi}\frac{\sin^2 u}{R + r\cos u}\mathrm{d}u$ だけ考えよう

$$\int_0^{2\pi}\frac{\sin^2 u}{R + r\cos u}\mathrm{d}u = \int_0^{2\pi}\frac{1 - \cos^2 u}{R + r\cos u}\mathrm{d}u \tag{7}$$
$$= \int_0^{2\pi}\frac{\frac{1}{r^2}(R + r\cos u)(R - r\cos u) + 1 - \frac{R^2}{r^2}}{R + r\cos u}\mathrm{d}u \tag{8}$$
$$= \int_0^{2\pi}\left(\frac{1}{r^2}(R - r\cos u) + \frac{1 - \frac{R^2}{r^2}}{R + r\cos u}\right)\mathrm{d}u \tag{9}$$
$$= \int_0^{2\pi}\left(\frac{R}{r^2} - \frac{1}{r}\cos u\right)\mathrm{d}u + \frac{r^2 - R^2}{r^2}\int_0^{2\pi}\frac{1}{R + r\cos u}\mathrm{d}u \tag{10}$$
$$= \frac{2\pi R}{r^2} + \frac{r^2 - R^2}{r^2}\int_0^{2\pi}\frac{1}{R + r\cos u}\mathrm{d}u \tag{11}$$

ここで、$\displaystyle\int_0^{2\pi}\frac{1}{R + r\cos u}\mathrm{d}u$ を計算しよう

$$\int_0^{2\pi}\frac{1}{R + r\cos u}\mathrm{d}u = \int_0^{2\pi}\frac{R - r\cos u}{R^2 - r^2\cos^2 u}\mathrm{d}u \tag{12}$$
$$= R\int_0^{2\pi}\frac{1}{R^2 - r^2\cos^2 u}\mathrm{d}u - r\int_0^{2\pi}\frac{\cos u}{R^2 - r^2\cos^2 u}\mathrm{d}u \tag{13}$$
$$= R\int_0^{2\pi}\frac{1}{R^2 - r^2\cos^2 u}\mathrm{d}u \tag{14}$$

後ろの $\displaystyle\int_0^{2\pi}\frac{\cos u}{R^2 - r^2\cos^2 u}\mathrm{d}u$ について、$\dfrac{\cos u}{R^2 - r^2\cos^2 u}$ は奇関数だから積分範囲内で総和 0

$$\int_0^{2\pi}\frac{1}{R^2 - r^2\cos^2 u}\mathrm{d}u = \int_0^{2\pi}\frac{1}{R^2 - r^2\frac{\cos 2u + 1}{2}}\mathrm{d}u \tag{15}$$
$$= \int_0^{2\pi}\frac{1}{\left(R^2 - \frac{r^2}{2}\right) - \frac{r^2}{2}\cos 2u}\mathrm{d}u \tag{16}$$

計算便利のため、$a = R^2 - \dfrac{r^2}{2}, b = \dfrac{r^2}{2}$ とすると (28) 式は $\displaystyle\int_0^{2\pi}\frac{1}{a - b\cos 2u}\mathrm{d}u$ になり.
また、定積分の処理が面倒なので、不定積分の形で計算しよう

$$\int\frac{1}{a - b\cos 2u}\mathrm{d}u = \frac{1}{2}\int\frac{1}{a - b\cos x}\mathrm{d}x \tag{17}$$

ここで、$t=\tan\dfrac{x}{2}$ と変換すると $\mathrm{d}x=\dfrac{2}{1+t^2}\mathrm{d}t, \cos x=\dfrac{1-t^2}{1+t^2}$ で
積分の上下限は共に 0 になるが、実際 2 回 $-\infty\to\infty$ の広義積分が出てくるから

$$\frac{1}{2}\int\frac{1}{a-b\cos x}\mathrm{d}x=2\int\frac{1}{2}\cdot\frac{1}{a-b\cdot\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\mathrm{d}t \tag{18}$$

$$=2\int\frac{1}{a\left(1+t^2\right)-b\left(1-t^2\right)}\mathrm{d}t \tag{19}$$

$$=2\int\frac{1}{(a-b)+(a+b)\,t^2}\mathrm{d}t \tag{20}$$

$$=\frac{2}{a-b}\int\frac{1}{1+\frac{a+b}{a-b}t^2}\mathrm{d}t \tag{21}$$

そして、$\dfrac{a+b}{a-b}t^2=\tan^2\theta$ となる変数変換をすると $\mathrm{d}t=\dfrac{1}{\cos^2\theta}\sqrt{\dfrac{a-b}{a+b}}\mathrm{d}\theta$ となり

$$\frac{2}{a-b}\int\frac{1}{1+\frac{a+b}{a-b}t^2}\mathrm{d}t=\frac{2}{a-b}\int\frac{1}{1+\tan^2\theta}\frac{1}{\cos^2\theta}\sqrt{\frac{a-b}{a+b}}\mathrm{d}\theta \tag{22}$$

$$=\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\int\mathrm{d}\theta \tag{23}$$

$$=\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\theta \tag{24}$$

$$=\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}t\right) \tag{25}$$

$$=\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}\tan\frac{x}{2}\right) \tag{26}$$

$$=\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}\tan u\right) \tag{27}$$

以上より

$$\int\frac{1}{a-b\cos 2u}\mathrm{d}u=\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}\tan u\right)+C \tag{28}$$

から

$$\int_0^{2\pi}\frac{1}{a-b\cos 2u}\mathrm{d}u=\lim_{\epsilon\to 0+}\left[\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}\tan u\right)\right]_0^{\frac{\pi}{2}-\epsilon} \tag{29}$$

$$+\lim_{\epsilon\to 0+}\left[\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}\tan u\right)\right]_{\frac{\pi}{2}+\epsilon}^{\frac{3\pi}{2}-\epsilon} \tag{30}$$

$$+\lim_{\epsilon\to 0+}\left[\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}\tan u\right)\right]_{\frac{3\pi}{2}+\epsilon}^{2\pi} \tag{31}$$

$$=\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\cdot\frac{\pi}{2}\cdot 2 \tag{32}$$

$$=\frac{2\pi}{a-b}\sqrt{\frac{a-b}{a+b}} \tag{33}$$

よって

$$\int_0^{2\pi} \frac{1}{R + r\cos u} \mathrm{d}u = R \int_0^{2\pi} \frac{1}{a - b\cos 2u} \mathrm{d}u \tag{34}$$

$$= \frac{2\pi R}{a-b}\sqrt{\frac{a-b}{a+b}} \tag{35}$$

$$= \frac{2\pi R}{R^2 - r^2}\sqrt{\frac{R^2 - r^2}{R^2}} \tag{36}$$

(24) 式に代入すると

$$\int_0^{2\pi} \frac{\sin^2 u}{R + r\cos u} \mathrm{d}u = \frac{2\pi R}{r^2} - \frac{R^2 - r^2}{r^2} \int_0^{2\pi} \frac{1}{R + r\cos u} \mathrm{d}u \tag{37}$$

$$= \frac{2\pi R}{r^2} - \frac{R^2 - r^2}{r^2} \frac{2\pi R}{R^2 - r^2}\sqrt{\frac{R^2 - r^2}{R^2}} \tag{38}$$

$$= \frac{2\pi R}{r^2} - \frac{2\pi R}{r^2}\sqrt{\frac{R^2 - r^2}{R^2}} \tag{39}$$

$$= \frac{2\pi R}{r^2} - \frac{2\pi}{r^2}\sqrt{R^2 - r^2} \tag{40}$$

$$= \frac{2\pi}{r^2}\left(R - \sqrt{R^2 - r^2}\right) \tag{41}$$

だから

$$\iint_S v \cdot \mathrm{d}A = -2\pi r^2 \int_0^{2\pi} \frac{\sin^2 u}{R + r\cos u} \mathrm{d}u \tag{42}$$

$$= -2\pi r^2 \cdot \frac{2\pi}{r^2}\left(R - \sqrt{R^2 - r^2}\right) \tag{43}$$

$$= -4\pi^2\left(R - \sqrt{R^2 - r^2}\right) \tag{44}$$

(iv)

$v = \nabla \times \omega$ をみたす ω が存在すると仮定すると、Stokes の定理より

$$\iint_S v \cdot \mathrm{d}A = \iint_S (\nabla \times \omega) \cdot \mathrm{d}A = 0 \tag{45}$$

(2) の計算結果により、$R = \sqrt{R^2 - r^2}$. 言い換えれば、$r = 0$
これは $r > 0$ と反するから. このような ω は存在しない