8.1

(1)

$$\partial V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R} \middle| \left(x^2+y^2+z^2=r^2\right) \vee \left(x^2+y^2+z^2=4r^2\right) \right\} = S^2(r) \cup S^2(2r)$$

(2)

単位法ベクトル場 ω は $\mathbf{p} \in S^2(a)$ に対し、$\omega(\mathbf{p}) = \dfrac{1}{a}\mathbf{p}$ となるので

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \partial V$ に対して $\mathbf{r}(x,y,z) = \begin{cases} \dfrac{1}{2r}\begin{pmatrix} x \\ y \\ z \end{pmatrix} & x^2+y^2+z^2=4r^2 \\ -\dfrac{1}{r}\begin{pmatrix} x \\ y \\ z \end{pmatrix} & x^2+y^2+z^2=r^2 \end{cases}$

(3)

$\nabla \cdot \mathbf{v} = 1+1+0=2$

(4)

$$\begin{aligned}
\iiint_{\overline{V}} (\nabla \cdot \mathbf{v})\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_r^{2r}\int_0^{\pi}\int_0^{2\pi} 2\rho^2 \sin\phi \,\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\rho \\
&= 4\pi \int_r^{2r}\int_0^{\pi} \rho^2 \sin\phi\,\mathrm{d}\phi\mathrm{d}\rho \\
&= 4\pi \int_r^{2r} \rho^2 \left[-\cos\phi\right]_0^{\pi} \mathrm{d}\rho \\
&= 8\pi \int_r^{2r} \rho^2 \mathrm{d}\rho \\
&= 8\pi \left[\frac{1}{3}\rho^3\right]_r^{2r} \\
&= \frac{56}{3}\pi r^3
\end{aligned}$$

$$\sigma\left(\rho', u, v\right) = \rho' \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix}, \sigma_u = \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ -\sin u \end{pmatrix}, \sigma_v = \begin{pmatrix} -\sin u \sin v \\ \sin u \cos v \\ 0 \end{pmatrix}$$

$$\begin{aligned} \sigma_u \times \sigma_v &= \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ -\sin u \end{pmatrix} \times \begin{pmatrix} -\sin u \sin v \\ \sin u \cos v \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \sin^2 u \cos v \\ \sin^2 u \sin v \\ \sin u \cos u \end{pmatrix} \\ &= \rho' \sin u \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \iint_{\partial V} \mathbf{v} \cdot \mathrm{d}\mathbf{A} &= \int_0^{\pi} \int_0^{2\pi} 2r \cdot 2r \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ 0 \end{pmatrix} \cdot 2r \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix} \sin u \mathrm{d}v \mathrm{d}u \\ &- \int_0^{\pi} \int_0^{2\pi} r \cdot r \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ 0 \end{pmatrix} \cdot r \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix} \sin u \mathrm{d}v \mathrm{d}u \\ &= \int_0^{\pi} \int_0^{2\pi} 7r^3 \sin^3 u \mathrm{d}v \mathrm{d}u \\ &= 14\pi r^3 \int_0^{\pi} \sin^3 u \mathrm{d}u \\ &= \frac{56}{3} \pi r^3 \end{aligned}$$

8.2

$$T=\left\{\left(\begin{array}{c} x \\ y \\ z \end{array}\right)\in\mathbb{R}^3 \middle| x\geq 0, y\geq 0, z\geq 0, x+y+z\leq 2\right\}, \nabla\cdot\mathbf{v}=2\left(x+y+z\right)$$

$$\begin{aligned}
\iint_{\partial T}\mathbf{v}\cdot\mathrm{d}\mathbf{A} &= \int_0^2\int_0^{2-x}\int_0^{2-x-y}2\left(x+y+z\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= 2\int_0^2\int_0^{2-x}\left[\left(x+y\right)z+\frac{1}{2}z^2\right]_0^{2-x-y}\mathrm{d}y\mathrm{d}x \\
&= 2\int_0^2\int_0^{2-x}\left(\left(x+y\right)\left(2-x-y\right)+\frac{1}{2}\left(2-x-y\right)^2\right)\mathrm{d}y\mathrm{d}x \\
&= \int_0^2\int_0^{2-x}\left(4-x^2-2xy-y^2\right)\mathrm{d}y\mathrm{d}x \\
&= \int_0^2\left[\left(4-x^2\right)y-xy^2-\frac{1}{3}y^3\right]_0^{2-x}\mathrm{d}x \\
&= \int_0^2\left(\left(2+x\right)\left(2-x\right)^2-x\left(2-x\right)^2-\frac{1}{3}\left(2-x\right)^3\right)\mathrm{d}x \\
&= \frac{1}{3}\int_0^2\left(x-2\right)^2\left(x+4\right)\mathrm{d}x \\
&= \frac{1}{3}\left[\frac{x^4}{4}-6x^2+16x\right]_0^2 \\
&= \frac{1}{3}\cdot 12=4
\end{aligned}$$

8.1

(1)

$$\partial V=\left\{\left(\begin{array}{c} x \\ y \\ z \end{array}\right)\in\mathbb{R}^3 \middle| \frac{x^2}{4}+y^2+z^2=1\right\}$$

(2)

$$f=\frac{x^2}{4}+y^2+z^2-1, \nabla f=\left(\begin{array}{c} \frac{x}{2} \\ 2y \\ 2z \end{array}\right)$$

$$\mathbf{n}=\frac{1}{\sqrt{\frac{x^2}{4}+y^2+z^2}}\left(\begin{array}{c} \frac{x}{2} \\ 2y \\ 2z \end{array}\right)$$

(3)

$$\nabla\cdot\mathbf{v}=1+1+1=3$$

(4)

$$\begin{aligned}\iint_{\partial V}(\nabla\cdot\mathbf{v})\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1\int_0^{2\pi}\int_0^{\pi} 3\rho^2\sin\phi\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}\rho\\ &= 6\int_0^1\int_0^{2\pi}\rho^2\mathrm{d}\theta\mathrm{d}\rho\\ &= 12\pi\int_0^1\rho^2\mathrm{d}\rho\\ &= 4\pi\end{aligned}$$

$$\sigma = \begin{pmatrix} 2\sin\phi\cos\theta\\ \sin\phi\sin\theta\\ \cos\phi\end{pmatrix}, \sigma_\phi = \begin{pmatrix} 2\cos\phi\cos\theta\\ \cos\phi\sin\theta\\ -\sin\phi\end{pmatrix}, \sigma_\theta = \begin{pmatrix} -2\sin\phi\sin\theta\\ \sin\phi\cos\theta\\ 0\end{pmatrix}$$

$$\begin{aligned}\sigma_\phi\times\sigma_\theta &= \begin{pmatrix} 2\cos\phi\cos\theta\\ \cos\phi\sin\theta\\ -\sin\phi\end{pmatrix}\times\begin{pmatrix} -2\sin\phi\sin\theta\\ \sin\phi\cos\theta\\ 0\end{pmatrix}\\ &= \begin{pmatrix}\sin^2\phi\cos\theta\\ 2\sin^2\phi\sin\theta\\ 2\sin\phi\cos\phi\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\iint_{\partial V}\mathbf{v}\cdot\mathrm{d}\mathbf{A} &= \int_0^{2\pi}\int_0^{\pi}\begin{pmatrix} 2\sin\phi\cos\theta\\ \sin\phi\sin\theta\\ \cos\phi\end{pmatrix}\cdot\begin{pmatrix}\sin^2\phi\cos\theta\\ 2\sin^2\phi\sin\theta\\ 2\sin\phi\cos\phi\end{pmatrix}\mathrm{d}\phi\mathrm{d}\theta\\ &= 2\int_0^{2\pi}\int_0^{\pi}\left(\sin^3\phi+\sin\phi\cos^2\phi\right)\mathrm{d}\phi\mathrm{d}\theta\\ &= 2\pi\int_0^{\pi}\sin\phi\mathrm{d}\theta\\ &= 4\pi\end{aligned}$$

8.2

$\nabla\cdot\mathbf{v} = y+0+2y = 3y$

$$\begin{aligned}\iint_{\partial T}\mathbf{v}\cdot\mathrm{d}\mathbf{A} &= \iiint_{\overline{T}} 3y\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &= \int_0^1\int_0^{1-x}\int_0^{1-x-y} 3y\mathrm{d}z\mathrm{d}y\mathrm{d}x\\ &= \int_0^1\int_0^{1-x} 3y\,(1-x-y)\,\mathrm{d}y\mathrm{d}x\\ &= \frac{1}{2}\int_0^1 (1-x)^3\,\mathrm{d}x\\ &= \frac{1}{8}\end{aligned}$$

8.3

$$\sigma = r\begin{pmatrix}\sin u\cos v\\ \sin u\sin v\\ \cos u\end{pmatrix}, \sigma_u = \begin{pmatrix}r\cos u\cos v\\ r\cos u\sin v\\ -r\sin u\end{pmatrix}, \sigma_v = \begin{pmatrix}-r\sin u\sin v\\ r\sin u\cos v\\ 0\end{pmatrix}$$

$$\begin{aligned}\sigma_u \times \sigma_v &= \begin{pmatrix}r\cos u\cos v\\ r\cos u\sin v\\ -r\sin u\end{pmatrix} \times \begin{pmatrix}-r\sin u\sin v\\ r\sin u\cos v\\ 0\end{pmatrix}\\ &= \begin{pmatrix}r^2\sin^2 u\cos v\\ r^2\sin^2 u\sin v\\ r^2\sin u\cos u\end{pmatrix}\end{aligned}$$

(1)

$$\begin{aligned}\iint_{S^2(r)} \mathbf{v}\cdot \mathrm{d}\mathbf{A} &= \iiint_{\overline{S^2(r)}} 3\left(x^2+y^2+z^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &= 3\int_0^r\int_0^{\sqrt{r^2-x^2}}\int_0^{\sqrt{r^2-x^2-y^2}}\left(x^2+y^2+z^2\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x\\ &= 3\int_0^r\int_0^{\sqrt{r^2-x^2}}\left(\left(x^2+y^2\right)\sqrt{r^2-x^2-y^2}+\frac{1}{3}\left(r^2-x^2-y^2\right)^{\frac{3}{2}}\right)\mathrm{d}y\mathrm{d}x\\ &= \frac{3}{8}\pi\int_0^r\left(r^4-x^4\right)\mathrm{d}x\\ &= \frac{3}{8}\pi\left[r^4x-\frac{1}{5}x^5\right]_0^r\\ &= \frac{3}{10}\pi r^5\end{aligned}$$

(2)

$$\begin{aligned}\iint_{S^2(r)} \mathbf{v}\cdot \mathrm{d}\mathbf{A} &= \iiint_{\overline{S^2(r)}} \left(x^2+y^2+z^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &= \frac{1}{3}\iiint_{\overline{S^2(r)}} 3\left(x^2+y^2+z^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &= \frac{1}{10}\pi r^5\end{aligned}$$

(3)

$\nabla \cdot \mathbf{v} = 3 + x + 2y$

$$
\begin{aligned}
\iint_{S^2(r)} \mathbf{v} \cdot \mathrm{d}\mathbf{A} &= \iiint_{\overline{S^2(r)}} (3 + x + 2y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} (3 + x + 2y)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \int_0^r \int_0^{\sqrt{r^2-x^2}} (3 + x + 2y)\sqrt{r^2 - x^2 - y^2}\mathrm{d}y\mathrm{d}x \\
&= \int_0^r \left(\frac{1}{4}\pi (x+3)\left(r^2 - x^2\right) + \frac{2}{3}\left(r^2 - x^2\right)^{\frac{3}{2}} \right) \mathrm{d}x \\
&= \frac{1}{16}\pi r^3 (3r + 8)
\end{aligned}
$$

8.4

$$
\sigma = \begin{pmatrix} (R + r\cos u)\cos v \\ (R + r\cos u)\sin v \\ r\sin u \end{pmatrix}, \sigma_u = \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix}, \sigma_v = \begin{pmatrix} -(R + r\cos u)\sin v \\ (R + r\cos u)\cos v \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
\sigma_u \times \sigma_v &= \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix} \times \begin{pmatrix} -(R + r\cos u)\sin v \\ (R + r\cos u)\cos v \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -r(R + r\cos u)\cos u\cos v \\ -r(R + r\cos u)\cos u\sin v \\ -r(R + r\cos u)\sin u \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\iint_{T_{R,r}} \mathbf{v} \cdot \mathrm{d}\mathbf{A} &= \iiint_{\overline{T_{R,r}}} 2\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= 2\int_0^r \int_0^{2\pi} \int_0^{2\pi} \rho(R + \rho\cos u)\,\mathrm{d}v\mathrm{d}u\mathrm{d}\rho \\
&= 4\pi \int_0^r \int_0^{2\pi} \rho(R + \rho\cos u)\,\mathrm{d}u\mathrm{d}\rho \\
&= 8\pi^2 R \int_0^r \rho\mathrm{d}\rho \\
&= 4\pi^2 R r^2
\end{aligned}
$$