9.1

(1)

$\mathbf{p} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} -3 \\ 2 \\ -2 \end{pmatrix}$

$$\begin{aligned}\alpha_{\mathbf{p}} &= 1 \cdot \mathrm{d}x \wedge \mathrm{d}y - 4 \cdot \mathrm{d}x \wedge \mathrm{d}z + 0 \cdot \mathrm{d}y \wedge \mathrm{d}z \\ &= \mathrm{d}x \wedge \mathrm{d}y - 4\mathrm{d}x \wedge \mathrm{d}z \\ &= \begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix} - 4 \begin{vmatrix} 1 & -3 \\ 3 & -2 \end{vmatrix} \\ &= 8 - 4 \cdot 7 = -20\end{aligned}$$

(2)

$\mathbf{p} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 6 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$

$$\begin{aligned}\omega_{\mathbf{p}} &= e^{-2} \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &= e^{-2} \begin{vmatrix} 3 & -1 & 4 \\ 2 & 6 & -1 \\ -1 & 4 & 2 \end{vmatrix}\end{aligned}$$

9.2

$\mathbf{p} = \begin{pmatrix} 1 \\ 0 \\ -3 \\ 2 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 6 \\ -1 \\ 2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -2 \\ -4 \\ 2 \\ 1 \end{pmatrix}$

$1 \leq i_1 \leq i_2 \leq i_3 \leq 4$ から、(i_1, i_2, i_3) が可能な組合は $(1,2,3), (1,2,4), (1,3,4), (2,3,4)$

$$\begin{aligned}\omega &= \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq 4} h_{i_1 i_2 i_3} \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \mathrm{d}x_{i_3} \\ &= h_{1,2,3} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + h_{1,2,4} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_4 \\ &+ h_{1,3,4} \mathrm{d}x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 + h_{2,3,4} \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \\ &= -2\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + 3\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_4 \\ &+ 0\mathrm{d}x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 - \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \\ &= -2 \begin{vmatrix} 1 & 2 & -2 \\ 2 & 6 & -4 \\ 1 & -1 & 2 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 & -2 \\ 2 & 6 & -4 \\ 2 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 6 & -4 \\ 1 & -1 & 2 \\ 2 & 2 & 1 \end{vmatrix} \\ &= -2 \cdot 8 + 3 \cdot 10 - (-8) \\ &= 22\end{aligned}$$

9.3

$$\begin{aligned}
f &= 3e^x \\
\alpha &= z^2\mathrm{d}x + 2\mathrm{d}y \\
\beta &= yz\mathrm{d}x + xz\mathrm{d}y + xy\mathrm{d}z \\
\omega &= z\mathrm{d}x \wedge \mathrm{d}y + y\mathrm{d}x \wedge \mathrm{d}z + x\mathrm{d}y \wedge \mathrm{d}z
\end{aligned}$$

$$\begin{aligned}
f \wedge \alpha &= 3e^x \left(z^2\mathrm{d}x + 2\mathrm{d}y\right) \\
&= 3e^x z^2\mathrm{d}x + 6e^x\mathrm{d}y
\end{aligned}$$

$$\alpha \wedge \alpha = 0$$

$$\begin{aligned}
\alpha \wedge \beta &= \left(z^2\mathrm{d}x + 2\mathrm{d}y\right) \wedge (yz\mathrm{d}x + xz\mathrm{d}y + xy\mathrm{d}z) \\
&= xz^3\mathrm{d}x \wedge \mathrm{d}y + xyz^2\mathrm{d}x \wedge \mathrm{d}z + 2yz\mathrm{d}y \wedge \mathrm{d}x + 2xy\mathrm{d}y \wedge \mathrm{d}z \\
&= \left(xz^3 - 2yz\right)\mathrm{d}x \wedge \mathrm{d}y + xyz^2\mathrm{d}x \wedge \mathrm{d}z + 2xy\mathrm{d}y \wedge \mathrm{d}z
\end{aligned}$$

$$\begin{aligned}
\beta \wedge \omega &= (yz\mathrm{d}x + xz\mathrm{d}y + xy\mathrm{d}z) \wedge (z\mathrm{d}x \wedge \mathrm{d}y + y\mathrm{d}x \wedge \mathrm{d}z + x\mathrm{d}y \wedge \mathrm{d}z) \\
&= xyz\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + xyz\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z + xyz\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= xyz\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z
\end{aligned}$$

9.4

$\omega = \sum_{1\le i_1<i_2\le 4} e^{x_{i_1}+x_{i_2}}\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} + \sum_{1\le i_2<i_1\le 4} e^{x_{i_1}+x_{i_2}}\mathrm{d}x_{i_2} \wedge \mathrm{d}x_{i_1} = 0$

$\omega \wedge \omega = 0 \wedge 0 = 0$

9.1

$$\begin{aligned}
\omega &= \sum_{1\le i_1<\cdots<i_k\le n} h_{i_1\cdots i_k}(\mathbf{p})\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}(\mathbf{v}_1, \cdots, \mathbf{v}_k) \\
&= \sum_{1\le i_1<\cdots<i_k\le n} k \begin{vmatrix} \mathrm{d}x_{i_1}(\mathbf{v}_1) & \cdots & \mathrm{d}x_{i_1}(\mathbf{v}_k) \\ \vdots & \ddots & \vdots \\ \mathrm{d}x_{i_k}(\mathbf{v}_1) & \cdots & \mathrm{d}x_{i_k}(\mathbf{v}_k) \end{vmatrix} \\
&= \sum_{1\le i_1<\cdots<i_k\le n} k \begin{vmatrix} \delta_{1,1} & \cdots & \delta_{k,1} \\ \vdots & \ddots & \vdots \\ \delta_{1,k} & \cdots & \delta_{k,k} \end{vmatrix} \\
&= \sum_{1\le i_1<\cdots<i_k\le n} k \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \\
&= \sum_{1\le i_1<\cdots<i_k\le n} k \\
&= k
\end{aligned}$$

9.2

$$\begin{aligned}
f &= 2e^{-yz} \\
\alpha &= xy\mathrm{d}y + e^x \mathrm{d}z \\
\beta &= y\mathrm{d}x + \mathrm{d}y - \sin x \mathrm{d}z \\
\omega &= x \sin x \mathrm{d}x \wedge \mathrm{d}y + \sin y \mathrm{d}x \wedge \mathrm{d}z + x \cos y \mathrm{d}y \wedge \mathrm{d}z
\end{aligned}$$

$$\begin{aligned}
f \wedge \alpha &= 2xye^{-yz}\mathrm{d}y + 2e^{x-yz}\mathrm{d}z \\
\alpha \wedge \beta &= (xy\mathrm{d}y + e^x\mathrm{d}z) \wedge (y\mathrm{d}x + \mathrm{d}y - \sin x\mathrm{d}z) \\
&= xy^2\mathrm{d}y \wedge \mathrm{d}x - xy \sin x \mathrm{d}y \wedge \mathrm{d}z + ye^x \mathrm{d}z \wedge \mathrm{d}x + e^x \mathrm{d}z \wedge \mathrm{d}y \\
\beta \wedge \omega &= (y\mathrm{d}x + \mathrm{d}y - \sin x\mathrm{d}z) \wedge (x \sin x \mathrm{d}x \wedge \mathrm{d}y + \sin y \mathrm{d}x \wedge \mathrm{d}z + x \cos y \mathrm{d}y \wedge \mathrm{d}z) \\
&= xy \cos y \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \sin y \mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z - x \sin^2 x \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(xy \cos y - \sin y - x \sin^2 x\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
\omega \wedge \omega &= 0
\end{aligned}$$

9.3

$$\begin{aligned}
\omega &= \sum_{1 \le j_1 < j_2 \le 4} g_{j_1 j_2} \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} + \sum_{1 \le j_2 < j_1 \le 4} g_{j_2 j_1} \mathrm{d}x_{j_2} \wedge \mathrm{d}x_{j_1} \\
&= \sum_{1 \le j_1 < j_2 \le 4} x_{j_1} x_{j_2} \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} + \sum_{1 \le j_2 < j_1 \le 4} x_{j_2} x_{j_1} \mathrm{d}x_{j_2} \wedge \mathrm{d}x_{j_1} \\
&= 0
\end{aligned}$$

から

$$\begin{aligned}
\alpha \wedge \omega &= \alpha \wedge 0 \\
&= 0
\end{aligned}$$