§10

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K9

(1)

(a)

$c_1(t) := \begin{pmatrix} b\cos t \\ b\sin t \end{pmatrix}, c_2(t) := \begin{pmatrix} a\cos t \\ -a\sin t \end{pmatrix}$

$$\begin{aligned}
\int_{\partial D} \mathbf{X}\cdot \mathrm{d}r &= \int_0^{2\pi} \begin{pmatrix} -b^3\sin t\cos^2 t \\ b^3\sin^2 t\cos t \end{pmatrix} \cdot \begin{pmatrix} -b\sin t \\ b\cos t \end{pmatrix} \mathrm{d}t + \int_0^{2\pi} \begin{pmatrix} a^3\sin t\cos^2 t \\ a^3\sin^2 t\cos t \end{pmatrix} \cdot \begin{pmatrix} -a\sin t \\ -a\cos t \end{pmatrix} \mathrm{d}t \\
&= \int_0^{2\pi} \left(b^4\sin^2 t\cos^2 t + b^4\sin^2 t\cos^2 t\right) \mathrm{d}t - \int_0^{2\pi} \left(a^4\sin^2 t\cos^2 t + a^4\sin^2 t\cos^2 t\right) \mathrm{d}t \\
&= 2b^4\int_0^{2\pi} \sin^2 t\cos^2 t\mathrm{d}t - 2a^4\int_0^{2\pi} \sin^2 t\cos^2 t\mathrm{d}t \\
&= \frac{\pi}{2}\left(b^4 - a^4\right)
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{\partial D} \mathbf{X}\cdot \mathrm{d}r &= \iint_D \nabla \times \mathbf{X}\mathrm{d}x_1\mathrm{d}x_2 \\
&= \iint_D \left(x_2^2 + x_1^2\right)\mathrm{d}x_1\mathrm{d}x_2 \\
&= \int_0^{2\pi}\int_a^b r^3\mathrm{d}r\mathrm{d}\theta \\
&= \frac{\pi}{2}\left(b^4 - a^4\right)
\end{aligned}$$

(2)

(a)

$$\begin{aligned}
\mathrm{rot}\mathbf{Y} &= \nabla \times \mathbf{Y} \\
&= \frac{\partial}{\partial x_1}\frac{x_1}{x_1^2 + x_2^2} + \frac{\partial}{\partial x_2}\frac{x_2}{x_1^2 + x_2^2} \\
&= \frac{x_2^2 - x_1^2}{\left(x_1^2 + x_2^2\right)^2} + \frac{x_1^2 - x_2^2}{\left(x_1^2 + x_2^2\right)^2} \\
&= 0
\end{aligned}$$

(b)

特異点は領域 D のうちに存在しないから、グリーン定理が使えるから

$$\begin{aligned}\int_c \mathbf{Y}\cdot \mathrm{d}r &= \iint_D (\nabla\times\mathbf{X})\,\mathrm{d}x_1\mathrm{d}x_2\\ &= \iint_D 0\mathrm{d}x_1\mathrm{d}x_2\\ &= 0\end{aligned}$$

(c)

$\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}, \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}$

$$\begin{aligned}\int_c \mathbf{Y}\cdot \mathrm{d}r &= \int_0^{2\pi} \begin{pmatrix} -\frac{\sin\theta}{r} \\ \frac{\cos\theta}{r} \end{pmatrix}\cdot\begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}\mathrm{d}\theta\\ &= \int_0^{2\pi}\mathrm{d}\theta\\ &= 2\pi\end{aligned}$$

P10.1

(1)

$$\begin{aligned}\int_c \mathbf{X}\cdot \mathrm{d}r &= \iint_D (\nabla\times\mathbf{X})\,\mathrm{d}x_1\mathrm{d}x_2\\ &= \iint_D (2x_2+2x_2)\,\mathrm{d}x_1\mathrm{d}x_2\\ &= 4\int_0^1\int_0^1 x_2\mathrm{d}x_1\mathrm{d}x_2\\ &= 2\end{aligned}$$

(2)

$$\begin{aligned}\int_c \mathbf{X}\cdot \mathrm{d}r &= \iint_D (\nabla\times\mathbf{X})\,\mathrm{d}x_1\mathrm{d}x_2\\ &= -\iint_D e^{x_1}x_2\mathrm{d}x_1\mathrm{d}x_2\\ &= -\int_0^{\pi}\int_0^{\sin x_1} e^{x_1}x_2\mathrm{d}x_2\mathrm{d}x_1\\ &= -\frac{1}{2}\int_0^{\pi} e^{x_1}\sin^2 x_1\mathrm{d}x_1\\ &= -\frac{1}{2}\cdot\frac{2}{5}\left(e^{\pi}-1\right)\\ &= -\frac{1}{5}\left(e^{\pi}-1\right)\end{aligned}$$

P10.2

(1)

$$
\begin{aligned}
S_D &= \iint_D \nabla \times \mathbf{X}_i \mathrm{d}x_1 \mathrm{d}x_2 \\
&= \begin{cases} \dfrac{1}{2} \iint_D \nabla \times \mathbf{X}_1 \mathrm{d}x_1 \mathrm{d}x_2 \\ \iint_D \nabla \times \mathbf{X}_1 \mathrm{d}x_1 \mathrm{d}x_2 \\ \iint_D \nabla \times \mathbf{X}_1 \mathrm{d}x_1 \mathrm{d}x_2 \end{cases} \\
&= \begin{cases} \dfrac{1}{2} \int_c \mathbf{X}_1 \cdot \mathrm{d}r \\ \int_c \mathbf{X}_2 \cdot \mathrm{d}r \\ \int_c \mathbf{X}_3 \cdot \mathrm{d}r \end{cases}
\end{aligned}
$$

(2)

(a)

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\theta \\ b\sin\theta \end{pmatrix}, \dfrac{\mathrm{d}}{\mathrm{d}\theta} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix}$

$\mathbf{X}_1 = \begin{pmatrix} -b\sin\theta \\ a\cos\theta \end{pmatrix}, \mathbf{X}_2 = \begin{pmatrix} 0 \\ a\cos\theta \end{pmatrix}, \mathbf{X}_3 = \begin{pmatrix} -b\sin\theta \\ 0 \end{pmatrix}$

$$
\begin{aligned}
\int_c \mathbf{X}_1 \cdot \mathrm{d}r &= \frac{1}{2} \int_0^{2\pi} \begin{pmatrix} -b\sin\theta \\ a\cos\theta \end{pmatrix} \cdot \begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix} \mathrm{d}\theta \\
&= \frac{ab}{2} \cdot 2\pi \\
&= \pi ab
\end{aligned}
$$

$$
\begin{aligned}
\int_c \mathbf{X}_2 \cdot \mathrm{d}r &= \int_0^{2\pi} \begin{pmatrix} 0 \\ a\cos\theta \end{pmatrix} \cdot \begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix} \mathrm{d}\theta \\
&= ab \int_0^{2\pi} \cos^2\theta \mathrm{d}\theta \\
&= \pi ab
\end{aligned}
$$

$$
\begin{aligned}
\int_c \mathbf{X}_3 \cdot \mathrm{d}r &= \int_0^{2\pi} \begin{pmatrix} -b\sin\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix} \mathrm{d}\theta \\
&= ab \int_0^{2\pi} \sin^2\theta \mathrm{d}\theta \\
&= \pi ab
\end{aligned}
$$

(b)

$c = \begin{pmatrix} a(t-\sin t) \\ a(1-\cos t) \end{pmatrix}, c' = \begin{pmatrix} a(1-\cos t) \\ a\sin t \end{pmatrix}$

$\mathbf{X}_1 = \begin{pmatrix} -a(1-\cos t) \\ a(t-\sin t) \end{pmatrix}, \mathbf{X}_2 = \begin{pmatrix} 0 \\ a(t-\sin t) \end{pmatrix}, \mathbf{X}_3 = \begin{pmatrix} -a(1-\cos t) \\ 0 \end{pmatrix}$

$$\begin{aligned}
\int_c \mathbf{X}_1 \cdot \mathrm{d}r &= \frac{1}{2}\int_0^{2\pi} \begin{pmatrix} -a(1-\cos t) \\ a(t-\sin t) \end{pmatrix} \cdot \begin{pmatrix} a(1-\cos t) \\ a\sin t \end{pmatrix} \mathrm{d}t \\
&= \frac{a^2}{2}\int_0^{2\pi} (t\sin t + 2\cos t - 2)\,\mathrm{d}t \\
&= -\frac{a^2}{2} \cdot 6\pi \\
&= -3a^2\pi
\end{aligned}$$

$$\begin{aligned}
\int_c \mathbf{X}_2 \cdot \mathrm{d}r &= \int_0^{2\pi} \begin{pmatrix} 0 \\ a(t-\sin t) \end{pmatrix} \cdot \begin{pmatrix} a(1-\cos t) \\ a\sin t \end{pmatrix} \mathrm{d}t \\
&= a^2\int_0^{2\pi} \left(t\sin t - \sin^2 t\right)\mathrm{d}t \\
&= -a^2 \cdot 3\pi \\
&= -3a^2\pi
\end{aligned}$$

$$\begin{aligned}
\int_c \mathbf{X}_3 \cdot \mathrm{d}r &= \int_0^{2\pi} \begin{pmatrix} -a(1-\cos t) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a(1-\cos t) \\ a\sin t \end{pmatrix} \mathrm{d}t \\
&= -a^2\int_0^{2\pi} \left(1 - \cos^2 t\right)\mathrm{d}t \\
&= -a^2 \cdot 3\pi \\
&= -3a^2\pi
\end{aligned}$$

(c)

$c = \begin{pmatrix} a\cos^3 t \\ a\sin^3 t \end{pmatrix}, c' = \begin{pmatrix} -3a\sin t\cos^2 t \\ 3a\sin^2 t\cos t \end{pmatrix}$

$\mathbf{X}_1 = \begin{pmatrix} -a\sin^3 t \\ a\cos^3 t \end{pmatrix}, \mathbf{X}_2 = \begin{pmatrix} 0 \\ a\cos^3 t \end{pmatrix}, \mathbf{X}_3 = \begin{pmatrix} -a\sin^3 t \\ 0 \end{pmatrix}$

$$\begin{aligned}
\int_c \mathbf{X}_1 \cdot \mathrm{d}r &= \frac{1}{2}\int_0^{2\pi} \begin{pmatrix} -a\sin^3 t \\ a\cos^3 t \end{pmatrix} \cdot \begin{pmatrix} -3a\sin t\cos^2 t \\ 3a\sin^2 t\cos t \end{pmatrix} \mathrm{d}t \\
&= \frac{3a^2}{2}\int_0^{2\pi} \left(\sin^4 t\cos^2 t + \sin^2 t\cos^4 t\right)\mathrm{d}t \\
&= \frac{3a^2}{2} \cdot \frac{\pi}{4} \\
&= \frac{3a^2\pi}{8}
\end{aligned}$$

$$
\begin{aligned}
\int_c \mathbf{X}_2 \cdot \mathrm{d}r &= \int_0^{2\pi} \begin{pmatrix} 0 \\ a\cos^3 t \end{pmatrix} \cdot \begin{pmatrix} -3a\sin t\cos^2 t \\ 3a\sin^2 t\cos t \end{pmatrix} \mathrm{d}t \\
&= 3a^2 \int_0^{2\pi} \sin^2 t \cos^4 t \mathrm{d}t \\
&= 3a^2 \cdot \frac{\pi}{8} \\
&= \frac{3a^2\pi}{8}
\end{aligned}
$$

$$
\begin{aligned}
\int_c \mathbf{X}_3 \cdot \mathrm{d}r &= \int_0^{2\pi} \begin{pmatrix} -a\sin^3 t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3a\sin t\cos^2 t \\ 3a\sin^2 t\cos t \end{pmatrix} \mathrm{d}t \\
&= 3a^2 \int_0^{2\pi} \sin^4 t \cos^2 t \mathrm{d}t \\
&= 3a^2 \cdot \frac{\pi}{8} \\
&= \frac{3a^2\pi}{8}
\end{aligned}
$$

P10.3

$$
\begin{aligned}
\mathrm{RHS} &= \int_c \mathbf{X} \cdot \mathbf{N} \mathrm{d}t \\
&= \int_c \mathbf{X} \cdot \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{T} \right) \mathrm{d}t \\
&= \int_c \mathbf{X} \cdot \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} c_1 \\ \frac{\mathrm{d}}{\mathrm{d}t} c_2 \end{pmatrix} \right) \mathrm{d}t \\
&= \int_c \mathbf{X} \cdot \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}t} c_2 \\ \frac{\mathrm{d}}{\mathrm{d}t} c_1 \end{pmatrix} \mathrm{d}t \\
&= \int_c \left(-X_1 \frac{\mathrm{d}}{\mathrm{d}t} c_2 + X_2 \frac{\mathrm{d}}{\mathrm{d}t} c_1 \right) \mathrm{d}t \\
&= \int_c \left(-X_1 \mathrm{d}c_2 + X_2 \mathrm{d}c_1 \right) \mathrm{d}t \\
&= \oint_{\partial D} \left(P \mathrm{d}c_1 + Q \mathrm{d}c_2 \right) (where\ P = X_2, Q = -X_1) \\
&= \iint_D \left(\frac{\partial Q}{\partial c_1} - \frac{\partial P}{\partial c_2} \right) \mathrm{d}x_1 \mathrm{d}x_2 \\
&= \iint_D \nabla \cdot \mathbf{X} \mathrm{d}t \\
&= \mathrm{LHS}
\end{aligned}
$$

P10.4

$\nabla \cdot (\mathbf{X} f) = (\nabla \cdot \mathbf{X}) f + \mathbf{X} \cdot \nabla f$
$\nabla \cdot \mathbf{X} = 0 \Longrightarrow \mathbf{X} \cdot \nabla f = \nabla \cdot (\mathbf{X} f)$

$$
\begin{aligned}
\iint_D \mathbf{X} \cdot \nabla f \mathrm{d}x_1 \mathrm{d}x_2 &= \iint_D \nabla \cdot (\mathbf{X} f) \, \mathrm{d}x_1 \mathrm{d}x_2 \\
&\overset{P10.3}{=} \int_c (\mathbf{X} f) \cdot \mathbf{N} \mathrm{d}t \\
&\overset{equipotential\ surface}{=} c \int_c \mathbf{X} \cdot \mathbf{N} \mathrm{d}t \\
&= c \iint_D \nabla \cdot \mathbf{X} \mathrm{d}x_1 \mathrm{d}x_2 \\
&\overset{\nabla \cdot \mathbf{X} = 0}{=} c \cdot 0 \\
&= 0
\end{aligned}
$$

P10.5

(1)

$$
\begin{aligned}
\frac{\partial F}{\partial r} &= \frac{\partial}{\partial r} \int_0^{2\pi} f\left(c_r\left(\theta\right)\right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{\partial}{\partial r} f\left(c_r\left(\theta\right)\right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \nabla f\left(c_r\left(\theta\right)\right) \cdot \frac{\partial}{\partial r} c_r\left(\theta\right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \nabla f\left(c_r\left(\theta\right)\right) \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \mathrm{d}\theta \\
&= \int_{c_r} \nabla f \cdot \left(-\mathbf{N}\right) \mathrm{d}t \\
&= \int_{c_r} \nabla f \cdot \mathbf{N} \mathrm{d}t
\end{aligned}
$$

最後の等号では、円の法ベクトルの定義は円心向きではなく、円外向きであるから、反時計回り 90° にした後反方向にすればいい

(2)

$$
\begin{aligned}
f\left(a\right) &= \lim_{r \to 0} F\left(r\right) \\
&= \lim_{r \to 0} \int_0^{2\pi} f\left(c_r\left(\theta\right)\right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \lim_{r \to 0} f\left(c_r\left(\theta\right)\right) \mathrm{d}\theta \\
&= \int_0^{2\pi} f\left(a\right) \mathrm{d}\theta \\
&= 2\pi f\left(a\right)
\end{aligned}
$$

両辺が等しくないから、帰一化因子 $\frac{1}{2\pi}$ が必要

i.e.$f\left(a\right) = \lim_{r \to 0} \frac{1}{2\pi} F\left(r\right) = \frac{1}{2\pi} \int_0^{2\pi} f\left(c_r\left(\theta\right)\right) \mathrm{d}\theta$

別解[1]

Lem. *Green 1:*
$\int_U (\nabla f\cdot\nabla g + f\Delta g)\,\mathrm{d}V = \int_{\partial U}(\nabla g\cdot\mathbf{N})\,f\mathrm{d}S$
Green 2:
$\int_U (f\Delta g - g\Delta f)\,\mathrm{d}V = \int_{\partial U}\left((\nabla g\cdot\mathbf{N})\,f - (\nabla f\cdot\mathbf{N})\,g\right)\mathrm{d}S$

Proof. Green 1 はもう証明したから略
Green 1 の f,g を交換して差をとると Green 2 が得られる □

Thm. f は $\overline{B(a,r)}$ での調和関数とする.

$$f(a) = \frac{1}{S(r)}\int_{\partial B(a,r)} f(x)\,\mathrm{d}S = \frac{1}{V(r)}\int_{B(a,r)} f(x)\,\mathrm{d}V$$

ここで、$S(r), V(r)$ はそれぞれ球の表面積と体積である

Proof. まず球面の状況を考える（次元 >2）
$U = B(0,r) - \overline{B(0,\epsilon)}, where \epsilon \in (0,r)$
$f, |x|^{2-n}$ に対して、Green 2 を使って

$$\begin{aligned}
0 &= \int_{\partial U}\left(f\mathrm{D}_N\left(|x|^{2-n}\right) - |x|^{2-n}\,\mathrm{D}_N f\right)\mathrm{d}S \\
&= \int_{\partial U} f\mathrm{D}_N\left(|x|^{2-n}\right)\mathrm{d}S \\
\int_{\partial B(0,r)} f\cdot(2-n)\,|x|^{1-n}\,\mathrm{d}S &= \int_{\partial B(0,\epsilon)} f\cdot(2-n)\,|x|^{1-n}\,\mathrm{d}S \\
\frac{1}{S(r)}\int_{\partial B(0,r)} f\mathrm{d}S &= \frac{1}{S(\epsilon)}\int_{\partial B(0,\epsilon)} f\mathrm{d}S \\
\xrightarrow{\epsilon\to 0}\ \frac{1}{S(r)}\int_{\partial B(0,r)} f\mathrm{d}S &= f(0)
\end{aligned}$$

$n=2$ のときは $|x|^{2-n}$ の代わりに $\log|x|$ を使えばいい
体積の場合は、$f, |x|^2$ に対して Green 2 を使えば $\nabla\left(|x|^2\right) = 2n, \mathrm{D}_N\left(|x|^2\right) = 2r$ を注意すると
よって、$\int_{B(0,r)} 2nf\cdot\mathrm{d}V = \int_{\partial B(0,r)} 2rf\cdot\mathrm{d}S$、$nV(r) = rS(r)$ と球面の場合を使えば得られる □

[1] GTM 137