§4
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K4.1

(i)

(a)

$|\mathbf{c}'(t)| = |(-r\sin t, r\cos t, a)| = \sqrt{r^2+a^2} > 0$から、正則曲線である

(b)

$$\begin{aligned} s(\mathbf{c}) &= \int_0^t \left|\mathbf{c}'(t)\right| \mathrm{d}t \\ &= \int_0^t \sqrt{r^2+a^2}\mathrm{d}t \\ &= t\sqrt{r^2+a^2} \end{aligned}$$

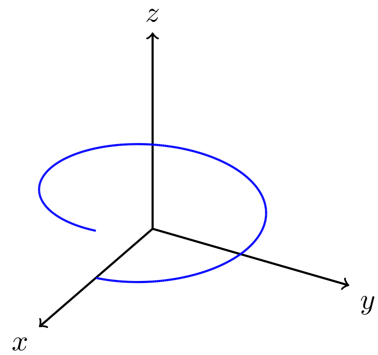
(c)

$t(s) = \dfrac{s}{\sqrt{r^2+a^2}}$で

$\mathbf{c}(s) = \left(r\cos\dfrac{s}{\sqrt{r^2+a^2}}, r\sin\dfrac{s}{\sqrt{r^2+a^2}}, \dfrac{as}{\sqrt{r^2+a^2}}\right)$

(d)

概形は以下のように



(ii)

(a)

$|\mathbf{c}'(t)| = \left|\left(1, \sinh\frac{t}{a}\right)\right| = \sqrt{1+\sinh^2\frac{t}{a}} = \cosh\frac{t}{a} \geq 1 > 0$
正則曲線である

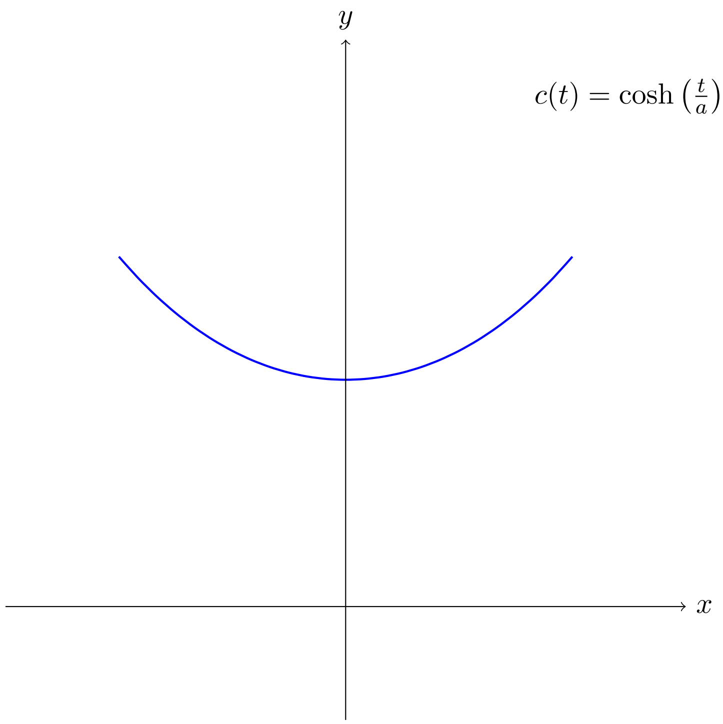
(b)

$$\begin{aligned} s(\mathbf{c}) &= \int_0^t \left|\mathbf{c}'(u)\right| \mathrm{d}u \\ &= \int_0^t \cosh\frac{u}{a}\mathrm{d}u \\ &= \left[a\sinh\frac{u}{a}\right]_0^t \\ &= a\sinh\frac{t}{a} \end{aligned}$$

(c)

$t(s) = a\sinh^{-1}\left(\frac{s}{a}\right)$で
$\mathbf{c}(s) = \left(a\sinh^{-1}\frac{s}{a}, \sqrt{s^2+a^2}\right)$

(d)



K4.2

(a)

$|\mathbf{c}'(t)| = |(-a\sin t, b\cos t)| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$
$a, b > 0$より、$\sqrt{a^2\sin^2 t + b^2\cos^2 t} > 0$
よって、$\mathbf{c}$は正則曲線である

(b)

$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = -a\sin t\mathrm{d}t$
$\mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t = b\cos t\mathrm{d}t$
$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{a^2\sin^2 t + b^2\cos^2 t}\mathrm{d}t$
i.e.$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$

解釈について、その弧長の微小変化は座標距離とみなせるから（なぜなら微小のときでは三角形に見なせる）あとは自然の計算と整理だけ

(c)

Chain's rule
$\mathbf{c}'(s) = \mathbf{c}'(t) \cdot \frac{\mathrm{d}t}{\mathrm{d}s} = (-a \sin t, b \cos t) \cdot \frac{1}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}$
$= \left(\frac{-a \sin t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}, \frac{b \cos t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} \right)$

P4.1

(1)

$\mathbf{c}$が正則曲線$\Rightarrow |\mathbf{c}'| \neq 0$
$\widetilde{\mathbf{c}}' = \mathbf{c}' \cdot \frac{\mathrm{d}t}{\mathrm{d}s}$で、$t(s)$は狭義単調増加であるから、$\frac{\mathrm{d}}{\mathrm{d}s} t(s) \neq 0$
よって、$\widetilde{\mathbf{c}}$も正則曲線である

(2)

$$\begin{aligned} L(\widetilde{\mathbf{c}}) &= \int_0^s \left|\widetilde{\mathbf{c}}'(u)\right| \mathrm{d}u \\ &= \int_0^t \left|\mathbf{c}'(v) \cdot \frac{\mathrm{d}v}{\mathrm{d}u}\right| \mathrm{d}u \\ &= \int_0^t \left|\mathbf{c}'(v)\right| \mathrm{d}v \\ &= L(\mathbf{c}) \end{aligned}$$

P4.2

(1)

(a)

$|\mathbf{c}'| = |(a(1-\cos t), a \sin t)| = \sqrt{a^2 \cos^2 t - 2a \cos t + a + a^2 \sin^2 t} = a\sqrt{2(1-\cos t)}$
よって、$\mathbf{c}$は正則曲線ではない
正則でない点は$x = 0, 2\pi$

(b)

$$
\begin{aligned}
L(\mathbf{c}) &= \int_0^{2\pi} \left|\mathbf{c}'(t)\right| \mathrm{d}t \\
&= \int_0^{2\pi} a\sqrt{2(1-\cos t)}\mathrm{d}t \\
&= \int_0^{2\pi} 2a\sin\frac{t}{2}\mathrm{d}t \\
&= 4a\left[-\cos\frac{t}{2}\right]_0^{2\pi} \\
&= 8a
\end{aligned}
$$

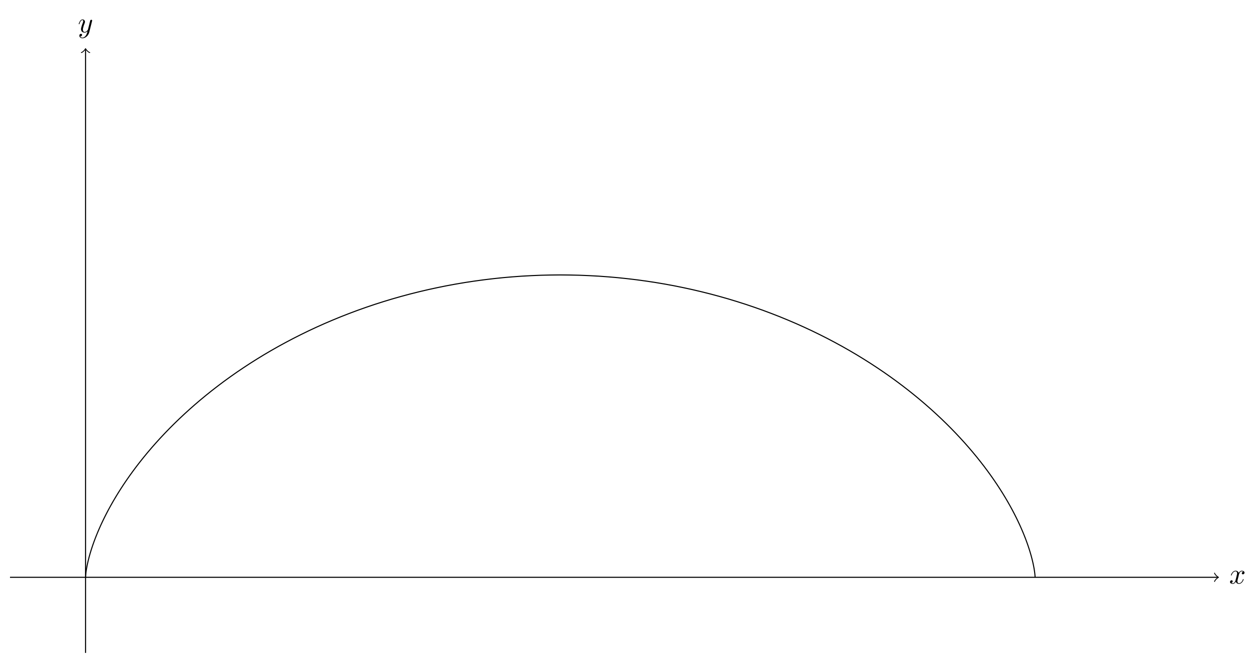
(c)

正則曲線でない

(d)

(a)より、正則でない点は$(0,0)$と$(2\pi a, 0)$

(e)



(2)

(a)

$|\mathbf{c}'(t)| = \left|\left(3a\cos^2 t\,(-\sin t)\,, 3a\sin^2 t\cos t\right)\right| = 3a\sin t\cos t$
正則でないのは$t = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$

(b)

$$\begin{aligned} L(\mathbf{c}) &= \int_0^{2\pi} \left|\mathbf{c}'(t)\right| \mathrm{d}t \\ &= \int_0^{2\pi} |3a\sin t\cos t|\, \mathrm{d}t \\ &= 12a \int_0^{\frac{\pi}{2}} \sin t\cos t \mathrm{d}t \\ &= 6a \end{aligned}$$

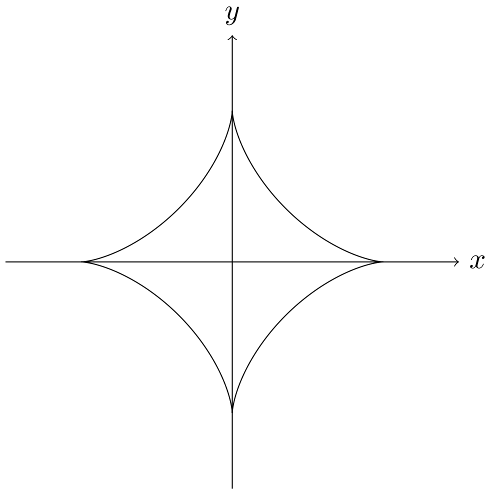
(c)

正則でない

(d)

$(a, 0)\,, (0, a)\,, (-a, 0)\,, (0, -a)$

(e)



P4.3

(1)

ここの$\mathbf{c}$というのは、極座標での曲線$r = r(\theta)$を**Descartes**座標に変換したものであるから
r正則$\Leftrightarrow$ $\mathbf{c}$正則

$$
\begin{aligned}
\left|\mathbf{c}'\right| &= \left|\left(r(\theta)\cos\theta, r(\theta)\sin\theta\right)'\right| \\
&= \left|\left(r'\cos\theta - r\sin\theta, r'\sin\theta + r\cos\theta\right)\right| \\
&= \sqrt{\left(r'^2\cos^2\theta - 2rr'\sin\theta\cos\theta + r^2\sin^2\theta\right) + \left(r'^2\sin^2\theta + 2rr'\sin\theta\cos\theta + r^2\cos^2\theta\right)} \\
&= \sqrt{r'^2(\theta) + r^2(\theta)}
\end{aligned}
$$

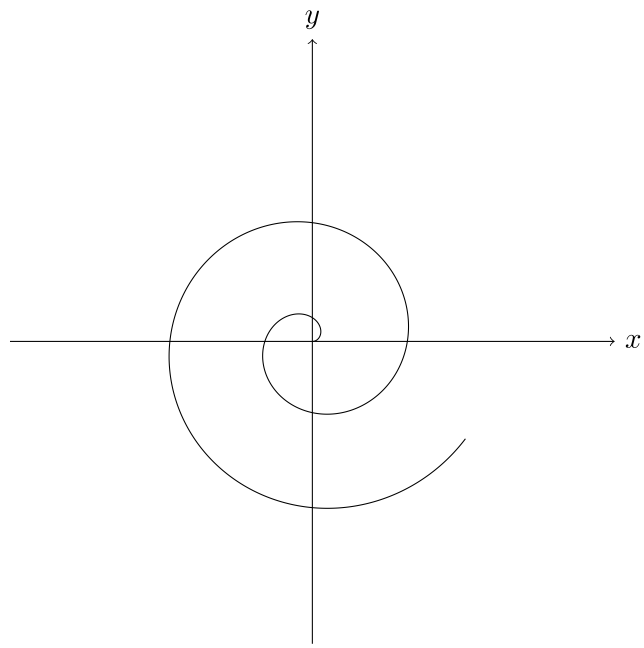
よって、$r(\theta)$は正則曲線である必要十分条件は$\sqrt{r'^2(\theta) + r^2(\theta)} \neq 0$

(2)

$$
\begin{aligned}
L(r) &= \int_0^\theta \sqrt{r'^2(u) + r^2(u)}\mathrm{d}u \\
&= \sqrt{r'^2(\theta) + r^2(\theta)}
\end{aligned}
$$

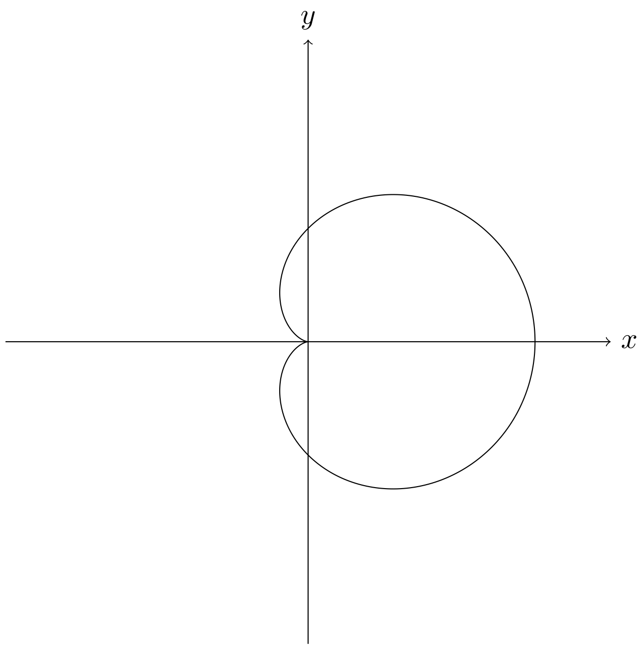
(3)

(a)



$$
\begin{aligned}
L(\mathbf{c}) &= \int_0^b \sqrt{a^2 + a^2\theta^2}\mathrm{d}\theta \\
&= a\int_0^b \sqrt{1+\theta^2}\mathrm{d}\theta \\
&= \frac{a}{2}\left(b\sqrt{b^2+1} + \sinh^{-1} b\right)
\end{aligned}
$$

(b)



$$
\begin{aligned}
L\left(\mathbf{c}\right) &= \int_0^{2\pi} \sqrt{a^2\sin^2\theta + a^2\cos^2\theta + 2a^2\cos\theta + a^2}\mathrm{d}\theta \\
&= a\int_0^{2\pi} \sqrt{2\left(1+\cos\theta\right)}\mathrm{d}\theta \\
&= 8a
\end{aligned}
$$