No.8
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K7

(1)

スカラー場だから

$c(t) = (1-t)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1-t \\ t \\ 1+t \end{pmatrix}$

$$\begin{aligned}\int_c f(x,y,z)\,\mathrm{d}t &= \int_0^1 (2+t)\cdot\sqrt{2}\mathrm{d}t \\ &= \sqrt{2}\left[\frac{1}{2}t^2 + 2t\right]_0^1 \\ &= \frac{5}{2}\sqrt{2}\end{aligned}$$

(2)

$c_1 = (1-t)\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2t-1 \\ 0 \end{pmatrix} \Longrightarrow c' = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$\mathbf{X} = \begin{pmatrix} (2t-1)^n \\ 0 \end{pmatrix} \Longrightarrow \mathbf{X}\cdot c' = 2(2t-1)^n$

$$\begin{aligned}\int_{c_1} &= 2\int_0^1 (2t-1)^n\,\mathrm{d}t \\ &= \int_{-1}^1 u^n\mathrm{d}u \\ &= \begin{cases} 0 & 2 \nmid n \\ \dfrac{2}{n+1} & 2|n \end{cases}\end{aligned}$$

$c_2 = \begin{pmatrix} t \\ \sqrt{1-t^2} \end{pmatrix}$,where $t \in [-1,1] \Longrightarrow c' = \begin{pmatrix} 1 \\ -\dfrac{t}{\sqrt{1-t^2}} \end{pmatrix}$

$\mathbf{X} = \begin{pmatrix} t^n \\ \left(1-t^2\right)^{\frac{n}{2}} \end{pmatrix} \Longrightarrow \mathbf{X}\cdot c' = t^n - \dfrac{t}{\sqrt{1-t^2}}\cdot\left(1-t^2\right)^{\frac{n}{2}} = t^n - t\left(1-t^2\right)^{\frac{n-1}{2}}$

$$\begin{aligned}\int_{c_2}\mathbf{X}\cdot\mathrm{d}r &= \int_{-1}^1\left(t^n - t\left(1-t^2\right)^{\frac{n-1}{2}}\right)\mathrm{d}t \\ &= \int_{-1}^1 t^n\mathrm{d}t \\ &= \begin{cases} 0 & 2 \nmid n \\ \dfrac{2}{n+1} & 2|n \end{cases}\end{aligned}$$

$$\begin{aligned}\int_{c_3} \mathbf{X} \cdot \mathrm{d}r &= \int_{c_1} \mathbf{X} \cdot \mathrm{d}r - \int_{c_2} \mathbf{X} \cdot \mathrm{d}r \\ &= \begin{cases} 0 & 2 \nmid n \\ 0 & 2 \mid n \end{cases}\end{aligned}$$

(3)

$$c_1 = (1-t)\begin{pmatrix} a \\ a \end{pmatrix} + t\begin{pmatrix} -a \\ a \end{pmatrix} = \begin{pmatrix} a-2at \\ a \end{pmatrix}$$
$$c_2 = (1-t)\begin{pmatrix} -a \\ a \end{pmatrix} + t\begin{pmatrix} -a \\ -a \end{pmatrix} = \begin{pmatrix} -a \\ a-2at \end{pmatrix}$$
$$c_3 = (1-t)\begin{pmatrix} -a \\ -a \end{pmatrix} + t\begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} 2at-a \\ 2at-a \end{pmatrix}$$
$$\mathbf{X} = \begin{cases} \begin{pmatrix} -a \\ a-2at \end{pmatrix} & c_1 \\ \begin{pmatrix} 2at-a \\ -a \end{pmatrix} & c_2 \\ \begin{pmatrix} a-2at \\ 2at-a \end{pmatrix} & c_3 \end{cases}$$
$$\mathbf{X} \cdot \mathrm{d}r = \begin{cases} 2a^2 & c_1 \\ 2a^2 & c_2 \\ 0 & c_3 \end{cases}$$

$$\begin{aligned}\int_c \mathbf{X} \cdot \mathrm{d}r &= \int_0^1 2a^2 \mathrm{d}t + \int_0^1 2a^2 \mathrm{d}t + \int_0^1 0 \mathrm{d}t \\ &= 4a^2\end{aligned}$$

P8.1

(1)

$$\begin{aligned}\int_c f(x,y,z)\, \mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1 \left(a^2\cos^2 t + a^2 \sin^2 t + b^2t^2\right) \cdot \sqrt{a^2+b^2}\mathrm{d}t \\ &= \sqrt{a^2+b^2}\int_0^1 \left(a^2 + b^2t^2\right)\mathrm{d}t \\ &= \sqrt{a^2+b^2}\left[a^2t + \frac{b^2}{3}t^3\right]_0^1 \\ &= \left(a^2 + \frac{b^3}{3}\right)\sqrt{a^2+b^2}\end{aligned}$$

(2)

$c = \begin{pmatrix} t \\ \frac{1}{2}t^2 \end{pmatrix}$,where $t \in [0,2]$

$c' = \begin{pmatrix} 1 \\ t \end{pmatrix}$

$$\begin{aligned}\int_c xy\mathrm{d}x\mathrm{d}y &= \int_0^2 \frac{1}{2}t^3 \cdot \sqrt{1+t^2}\mathrm{d}t \\ &= \frac{1}{2}\int_0^2 t^3\sqrt{1+t^2}\mathrm{d}t \\ &= \frac{1}{4}\int_0^4 u\sqrt{u+1}\mathrm{d}u \\ &= \frac{1}{15}\left(1+25\sqrt{5}\right)\end{aligned}$$

(3)

$c = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$, where $t \in [0, 2\pi]$

$c' = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ $\mathbf{X} \cdot \mathrm{d}r = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = 0$

$$\begin{aligned}\int_c \mathbf{X} \cdot \mathrm{d}r &= \int_0^{2\pi} 0\mathrm{d}t \\ &= 0\end{aligned}$$

(4)

$c = \begin{pmatrix} a\cos t \\ a\sin t \\ a^2\cos 2t \end{pmatrix}$

$c' = \begin{pmatrix} -a\sin t \\ a\cos t \\ -2a^2\sin 2t \end{pmatrix}$

$\mathbf{X} \cdot \mathrm{d}r = \begin{pmatrix} a^2\cos 2t \\ a\cos t \\ a\sin t \end{pmatrix} \cdot \begin{pmatrix} -a\sin t \\ a\cos t \\ -2a^2\sin 2t \end{pmatrix} = -a^3\sin t\cos 2t + a^2\cos^2 t - 2a^3\sin t\sin 2t$

$$\begin{aligned}\int_c \mathbf{X} \cdot \mathrm{d}r &= \int_0^{2\pi} \left(-a^3\sin t\cos 2t + a^2\cos^2 t - 2a^3\sin t\sin 2t\right)\mathrm{d}t \\ &= -a^3\int_0^{2\pi} \sin t\cos 2t\mathrm{d}t + a^2\int_0^{2\pi} \cos^2 t\mathrm{d}t - 2a^3\int_0^{2\pi} \sin t\sin 2t\mathrm{d}t \\ &= a^2\int_0^{2\pi} \cos^2 t\mathrm{d}t \\ &= \pi a^2\end{aligned}$$

(5)

(a)

$c_1 = \begin{pmatrix} t \\ t \\ t \end{pmatrix}, t \in [0,1], c' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\begin{aligned}\int_{c_1} \mathbf{X} \cdot \mathrm{d}r &= \int_0^1 \left(t + t^2 + t^3\right) \mathrm{d}t \\ &= \left[\frac{1}{4}t^4 + \frac{1}{3}t^3 + \frac{1}{2}t^2\right]_0^1 \\ &= \frac{13}{12}\end{aligned}$$

(b)

$c_{2_1} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}, c_{2_2} = \begin{pmatrix} 1 \\ t \\ 0 \end{pmatrix}, c_{2_3} = \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}$
$c'_{2_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, c'_{2_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, c'_{2_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{aligned}\int_c \mathbf{X} \cdot \mathrm{d}r &= \int_0^1 t^2 \mathrm{d}t + \int_0^1 t\mathrm{d}t + \int_0^1 t\mathrm{d}t \\ &= \frac{1}{3} + \frac{1}{2} + \frac{1}{2} \\ &= \frac{4}{3}\end{aligned}$$

P8.2

$$\begin{aligned}\int_{c\circ\phi} f\mathrm{d}s &= \int_\alpha^\beta f\left(c \circ \phi\left(t\right)\right) \left|\left(c \circ \phi\right)'\left(t\right)\right| \mathrm{d}t \\ &= \int_\alpha^\beta f\left(c \circ \phi\left(t\right)\right) \left|c'\left(\phi\left(t\right)\right)\right| \phi'\left(t\right) \mathrm{d}t \\ &= \int_{\phi^{-1}(\alpha)}^{\phi^{-1}(\beta)} f\left(c\left(t\right)\right) \left|c'\left(t\right)\right| \mathrm{d}t \\ &= \int_c f\mathrm{d}s\end{aligned}$$

$$\begin{aligned}\int_{c\circ\phi} \mathbf{X}\mathrm{d}r &= \int_\alpha^\beta \mathbf{X}\left(c \circ \phi\left(t\right)\right) \cdot \left(c \circ \phi\left(t\right)\right)' \mathrm{d}t \\ &= \int_\alpha^\beta \mathbf{X}\left(c \circ \phi\left(t\right)\right) \cdot c'\left(\phi\left(t\right)\right) \phi'\left(t\right) \mathrm{d}t \\ &= \int_{\phi^{-1}(\alpha)}^{\phi^{-1}(\beta)} \mathbf{X}\left(c\left(t\right)\right) \cdot c'\left(t\right) \mathrm{d}t \\ &= \int_c \mathbf{X}\mathrm{d}r\end{aligned}$$

P8.3

$$\begin{aligned}\left|\int_c \mathbf{X}\cdot \mathrm{d}r\right| &= \left|\int_a^b \mathbf{X}\left(c\left(t\right)\right)\cdot c'\left(t\right)\mathrm{d}t\right|\\ &\leq \int_a^b \left|\mathbf{X}\left(c\left(t\right)\right)\cdot c'\left(t\right)\right|\mathrm{d}t\\ &\leq \int_a^b \left|\mathbf{X}\left(c\left(t\right)\right)\right|\cdot\left|c'\left(t\right)\right|\mathrm{d}t\\ &\leq \left\|\mathbf{X}\right\|\int_a^b \left|c'\left(t\right)\right|\mathrm{d}t\\ &= \left\|\mathbf{X}\right\|\mathrm{L}\left(c\right)\end{aligned}$$

参考文献