§9

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K8

(1)

(a)

$$\begin{aligned}\nabla f &= \nabla(\mathbf{a}\cdot\mathbf{x})\\ &= \mathbf{a}\cdot\nabla(\mathbf{x})\\ &= \mathbf{a}\cdot 1\\ &= \mathbf{a}\end{aligned}$$

(b)

$\{\mathbf{x}\in\mathbb{R}^n | \mathbf{a}\cdot\mathbf{x}=k, k\in\mathbb{R}^n\}$
これは超平面（$n-1$ 次元のアフィン空間）である

(2)

$$\begin{aligned}\mathrm{rot}(\mathbf{X}) &= \nabla\times\mathbf{X}\\ &= \begin{pmatrix}\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2}\end{pmatrix}\times\mathbf{X}\\ &\overset{(a)}{=} \begin{pmatrix}\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2}\end{pmatrix}\times\left(\frac{f_2}{f_1^2+f_2^2}\nabla f_1-\frac{f_1}{f_1^2+f_2^2}\nabla f_2\right)\\ &= \begin{pmatrix}\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2}\end{pmatrix}\times\left(\nabla\left(\frac{f_2}{f_1^2+f_2^2}f_1-\frac{f_1}{f_1^2+f_2^2}f_2\right)\right)\\ &= \begin{pmatrix}\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2}\end{pmatrix}\times(\nabla\mathbf{0})\\ &= \begin{pmatrix}\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2}\end{pmatrix}\times\mathbf{0}\\ &= \mathbf{0}\end{aligned}$$

(3)

(a)

$$
\begin{aligned}
\nabla \times \mathbf{X} &= \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \end{pmatrix} \times \begin{pmatrix} g\left(\sqrt{x_1^2 + x_2{}^2}\right) g\left(x_1^r\right) \\ g\left(\sqrt{x_1^2 + x_2^2}\right) g\left(x_2^r\right) \end{pmatrix} \\
&= \frac{\partial}{\partial x_1}\left(g\left(\sqrt{x_1^2 + x_2^2}\right) g\left(x_2^r\right)\right) - \frac{\partial}{\partial x_2}\left(g\left(\sqrt{x_1^2 + x_2{}^2}\right) g\left(x_1^r\right)\right) \\
&= \frac{\partial}{\partial x_1} g\left(\sqrt{x_1^2 + x_2^2}\right) g\left(x_2^r\right) - \frac{\partial}{\partial x_2} g\left(\sqrt{x_1^2 + x_2^2}\right) g\left(x_1^r\right) \\
&= \frac{x_1 g\left(x_2^r\right)}{\sqrt{x_1^2 + y_2^2}} \frac{\partial g}{\partial x_1}\left(\sqrt{x_1^2 + x_2^2}\right) - \frac{x_2 g\left(x_1^r\right)}{\sqrt{x_1^2 + y_2^2}} \frac{\partial g}{\partial x_2}\left(\sqrt{x_1^2 + x_2^2}\right) \\
&= 0
\end{aligned}
$$

$\Longrightarrow$

$$
\begin{aligned}
\frac{x_1 g\left(x_2^r\right)}{\sqrt{x_1^2 + y_2^2}} \frac{\partial g}{\partial x_1}\left(\sqrt{x_1^2 + x_2^2}\right) &= \frac{x_2 g\left(x_1^r\right)}{\sqrt{x_1^2 + y_2^2}} \frac{\partial g}{\partial x_2}\left(\sqrt{x_1^2 + x_2^2}\right) \\
x_1 g\left(x_2^r\right) \frac{\partial g}{\partial x_1}\left(\sqrt{x_1^2 + x_2^2}\right) &= x_2 g\left(x_1^r\right) \frac{\partial g}{\partial x_2}\left(\sqrt{x_1^2 + x_2^2}\right) \\
x_1 g\left(x_2^r\right) g'\left(\sqrt{x_1^2 + x_2^2}\right) &= x_2 g\left(x_1^r\right) g'\left(\sqrt{x_1^2 + x_2^2}\right)
\end{aligned}
$$

$$
\Longrightarrow g'\left(\sqrt{x_1^2 + x_2^2}\right)\left(x_1 g\left(x_2^r\right) - x_2 g\left(x_1^r\right)\right) = 0
$$

$g'\left(\sqrt{x_1^2 + x_2^2}\right) \neq 0$ なら $g\left(x\right) = x^{\frac{1}{r}}$
$g'\left(\sqrt{x_1^2 + x_2^2}\right) = 0$ なら $g\left(x\right) = k \in \mathbb{R}$、定数関数てある

(b)

$g\left(x\right) = x^{\frac{1}{r}}$

$$
\mathbf{X} = \begin{pmatrix} \left(x_1^2 + x_2^2\right)^{\frac{1}{2r}} x_1 \\ \left(x_1^2 + x_2^2\right)^{\frac{1}{2r}} x_2 \end{pmatrix}
$$

$$
\begin{aligned}
f &= \begin{pmatrix} \displaystyle\int \left(x_1^2 + x_2^2\right)^{\frac{1}{2r}} x_1 \mathrm{d}x_1 \\ \displaystyle\int \left(x_1^2 + x_2^2\right)^{\frac{1}{2r}} x_2 \mathrm{d}x_2 \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{1}{2 + \frac{1}{r}} \left(x^2 + y^2\right)^{1 + \frac{1}{2r}} + C \\ \dfrac{1}{2 + \frac{1}{r}} \left(x^2 + y^2\right)^{1 + \frac{1}{2r}} + C \end{pmatrix}
\end{aligned}
$$

$$
f = \begin{pmatrix} \dfrac{1}{2 + \frac{1}{r}} \left(x^2 + y^2\right)^{1 + \frac{1}{2r}} + C \\ \dfrac{1}{2 + \frac{1}{r}} \left(x^2 + y^2\right)^{1 + \frac{1}{2r}} + C \end{pmatrix}
$$

$g(x) = k \in \mathbb{R}$

$$\mathbf{X} = \begin{pmatrix} k^2 \\ k^2 \end{pmatrix}$$

$$\begin{aligned} f &= \begin{pmatrix} \displaystyle\int k^2 \mathrm{d}x_1 \\ \displaystyle\int k^2 \mathrm{d}x_2 \end{pmatrix} \\ &= \begin{pmatrix} k^2 x_1 + C \\ k^2 x_2 + C \end{pmatrix} \end{aligned}$$

$f = \begin{pmatrix} k^2 x_1 + C \\ k^2 x_2 + C \end{pmatrix}$

P9.1

(1)

(a)

$$\begin{aligned} \mathbf{v}(af + bg) &= \frac{\mathrm{d}(af + bg)}{\mathrm{d}t}(\mathbf{x} + t\mathbf{v})\Big|_{t=0} \\ &= \frac{\mathrm{d}af}{\mathrm{d}t}(\mathbf{x} + t\mathbf{v})\Big|_{t=0} + \frac{\mathrm{d}bg}{\mathrm{d}t}(\mathbf{x} + t\mathbf{v})\Big|_{t=0} \\ &= a\frac{\mathrm{d}f}{\mathrm{d}t}(\mathbf{x} + t\mathbf{v})\Big|_{t=0} + b\frac{\mathrm{d}g}{\mathrm{d}t}(\mathbf{x} + t\mathbf{v})\Big|_{t=0} \\ &= a\mathbf{v}(f) + b\mathbf{v}(g) \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{v}(fg) &= \frac{\mathrm{d}fg}{\mathrm{d}t}(\mathbf{x} + t\mathbf{v})\Big|_{t=0} \\ &= \frac{\mathrm{d}f}{\mathrm{d}t}g(\mathbf{x} + t\mathbf{v})\Big|_{t=0} + f\frac{\mathrm{d}g}{\mathrm{d}t}(\mathbf{x} + t\mathbf{v})\Big|_{t=0} \\ &= \mathbf{v}(f)g + f\mathbf{v}(g) \end{aligned}$$

(2)

$$\begin{aligned} \mathbf{e}(f) &= \frac{\mathrm{d}f}{\mathrm{d}t}(\mathbf{x} + t\mathbf{e})\Big|_{t=0} \\ &= \nabla f \cdot \mathbf{e} \end{aligned}$$

$$\begin{aligned} |\mathbf{e}(f)| &= |\nabla f \cdot \mathbf{e}| \\ &\overset{Cauchy}{\leq} |\nabla f|\,|\mathbf{e}| \\ &= |\nabla f| \end{aligned}$$

よって、*Cauchy* の不等式の等号条件より明らかに平行する場合しか成り立たない（そうしないと $\mathbf{e}$ の射影は 1 より小さい）

P9.2

(1)

$$
\begin{aligned}
f(x_1, x_2) &= c \\
f(x_1, g(x_1)) &= c \\
\frac{\mathrm{d}}{\mathrm{d}x_1} f(x_1, g(x_1)) &= 0 \\
\frac{\partial}{\partial x_1} f(x_1, g(x_1)) + \frac{\partial}{\partial x_2} f(x_1, g(x_1)) \cdot \frac{\partial}{\partial x_1} g(x_1) &= 0 \\
\overset{\mathbf{x}_0=(a_1,a_2)}{\Longrightarrow} \frac{\partial}{\partial x_1} f(a_1, g(a_1)) + \frac{\partial}{\partial x_2} f(a_1, g(a_1)) \cdot \frac{\partial}{\partial x_1} g(a_1) &= 0 \\
\frac{\partial}{\partial x_1} g(a_1) &= -\frac{\frac{\partial}{\partial x_1} f(\mathbf{x}_0)}{\frac{\partial}{\partial x_2} f(\mathbf{x}_0)}
\end{aligned}
$$

(2)

点傾式で書くと

$$
\begin{aligned}
x_2 - a_2 &= \frac{\partial}{\partial x_1} g(a_1)(x_1 - a_1) \\
x_2 - a_2 &= -\frac{\frac{\partial}{\partial x_1} f(\mathbf{x}_0)}{\frac{\partial}{\partial x_2} f(\mathbf{x}_0)}(x_1 - a_1)
\end{aligned}
$$

$$
\frac{\partial}{\partial x_1} f(\mathbf{x}_0)(x_1 - a_1) + \frac{\partial}{\partial x_2} f(\mathbf{x}_0)(x_2 - a_2) = 0
$$

(a_1, a_2) について

$$
\begin{aligned}
\begin{pmatrix} \frac{\partial}{\partial x_1} f(\mathbf{x}_0) \\ \frac{\partial}{\partial x_2} f(\mathbf{x}_0) \end{pmatrix} \cdot \begin{pmatrix} a_1 - a_1 \\ a_2 - a_2 \end{pmatrix} &= \begin{pmatrix} \frac{\partial}{\partial x_1} f(\mathbf{x}_0) \\ \frac{\partial}{\partial x_2} f(\mathbf{x}_0) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&= 0
\end{aligned}
$$

これより、勾配ベクトル場と等位面が直交している

P9.3

$\Longrightarrow$

$$
\int_c \mathbf{X} \cdot \mathrm{d}\mathbf{r} \overset{\mathbf{X}=\nabla f}{=} f(b) - f(a)
$$

自明である

$\Longleftarrow$

$\displaystyle\int_c \mathbf{X}\cdot \mathrm{d}\mathbf{r}$ は端点しか依存しないから

$$\begin{aligned}\int_c \mathbf{X}\cdot \mathrm{d}\mathbf{r} &= \int_c \nabla f\cdot \mathrm{d}\mathbf{r}\\ &= f\left(c\left(b\right)\right)-f\left(c\left(a\right)\right)\end{aligned}$$

言い換えれば、$p_0, p \in [a,b]$ で

$$f\left(p\right)=\int_{p_0}^{p}\mathbf{X}\cdot \mathrm{d}\mathbf{r}$$

を逆に考えるとある $\mathbf{X}=\nabla f$ を満たす f が存在
これは (a) である

P9.4

$\Longrightarrow$

$$\begin{aligned}\nabla\times\mathbf{X} &= \nabla\times\nabla f\\ &= \mathbf{0}\end{aligned}$$

$\Longleftarrow$

以下は Stokes を黙認して使おう

$$\begin{aligned}\int_c \mathbf{X}\cdot \mathrm{d}\mathbf{r} &= \iint_S \left(\nabla\times\mathbf{X}\right)\cdot \mathrm{d}S\\ &= \iint_S 0\cdot \mathrm{d}S\\ &= 0\end{aligned}$$

言い換えれば、閉じた曲線の線積分が 0 であって、このベクトル場は conservative なベクトル場である
定義より、あるスカラー場 f が存在し、$s.t.\mathbf{X}=\nabla f$

P9.5

(1)

対称性を考えると、中心となれる点は $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ しかないが、その点は D に存在しない
また、逆に $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ と $\begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$ の線分では $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ が D に存在しない
よって、D は星状領域ではない

(2)

$$
\begin{aligned}
\nabla \times \mathbf{X} &= \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \end{pmatrix} \times \begin{pmatrix} -\dfrac{x_2}{x_1^2 + x_2^2} \\ \dfrac{x_1}{x_1^2 + x_2^2} \end{pmatrix} \\
&= \frac{\partial}{\partial x_1} \frac{x_1}{x_1^2 + x_2^2} + \frac{\partial}{\partial x_2} \frac{x_2}{x_1^2 + x_2^2} \\
&= -\frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} + \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} \\
&= \mathbf{0}
\end{aligned}
$$

(3)

$$
\begin{aligned}
\int_c \mathbf{X} \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \mathrm{d}t \\
&= 2\pi
\end{aligned}
$$

(4)

P9.4 の証明より、$\mathbf{X} = \nabla f$ が存在するなら、(2) と (3) の計算結果は同時に 0 になるはずであるが、その閉曲線の線積分は 0 ではない
よって、存在しない

参考文献