1 1.1

$$\begin{aligned}\int_C (2x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y &= \int_0^1 \left(2t - t^2\right)\mathrm{d}t + \left(t+t^2\right)2t\mathrm{d}t \\ &= \int_0^1 \left(2t^3 + t^2 + 2t\right)\mathrm{d}t \\ &= \left[\frac{1}{2}t^4 + \frac{1}{3}t^3 + t^2\right]_0^1 \\ &= \frac{11}{6}\end{aligned}$$

2 1.2

$$\begin{aligned}\int_C \mathbf{v}\cdot\mathrm{d}\mathbf{x} &= \int_0^1 \begin{pmatrix} 2(a+t)(b+t)c \\ (a+t)^2 c \\ (a+t)^2 (b+t)\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}\mathrm{d}t + \int_1^2 \begin{pmatrix} 2(a+1)(b+1)(c-1+t) \\ (a+1)^2(c-1+t) \\ (a+1)^2(b+1)\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\mathrm{d}t \\ &= \int_0^1 \left(2(a+t)(b+t)c + (a+t)^2 c\right)\mathrm{d}t + \int_1^2 (a+1)^2(b+1)\,\mathrm{d}t \\ &= c\left(a^2 + 2a(b+1) + b + 1\right) + (a+1)^2(b+1) \\ &= a^2(b+c+1) + (2a+1)(b+1)(c+1)\end{aligned}$$

また、線積分の性質を考えると

$$\begin{aligned}\int_C \mathbf{v}\cdot\mathrm{d}\mathbf{x} &= \int_C \nabla\left(x^2yz\right)\mathrm{d}\mathbf{x} \\ &= (a+1)^2(b+1)c - a^2bc + (a+1)^2(b+1)(c+1) - (a+1)^2(b+1)c \\ &= (a+1)^2(b+1)(c+1) - a^2bc\end{aligned}$$

3 1.3

$$\mathbf{v} = \begin{pmatrix} y-x \\ 3x+2y\end{pmatrix}, D := \left\{\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2 \middle| 0\le x, y\le 1, y\le x\right\}$$

$$\begin{aligned}\int_C (y-x)\,\mathrm{d}x + (3x+2y)\,\mathrm{d}y &= \iint_D \nabla\times\begin{pmatrix} y-x \\ 3x+2y\end{pmatrix}\mathrm{d}x\mathrm{d}y \\ &= \iint_D 2\mathrm{d}x\mathrm{d}y \\ &= \int_0^1\int_0^x 2\mathrm{d}y\mathrm{d}x \\ &= \int_0^1 2x\mathrm{d}x \\ &= 1\end{aligned}$$

4　1.4

(i)

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)&=\frac{-x^2+y^2}{\left(x^2+y^2\right)^2}-\frac{-x^2+y^2}{\left(x^2+y^2\right)^2}\\
&=0
\end{aligned}
$$

(ii)

場合分けにして、$\begin{pmatrix}0\\0\end{pmatrix}$ は閉曲線の内部に存在しないとき

$$
\begin{aligned}
\int_C\frac{-y}{x^2+y^2}\mathrm{d}x+\frac{x}{x^2+y^2}\mathrm{d}y&=\iint_D 0\mathrm{d}x\mathrm{d}y\\
&=0
\end{aligned}
$$

$\begin{pmatrix}0\\0\end{pmatrix}$ が内部に存在するとき、計算便利のため $C=\begin{pmatrix}r\cos\theta\\r\sin\theta\end{pmatrix}$ とする.$C'=\begin{pmatrix}-r\sin\theta\\r\cos\theta\end{pmatrix}$

$$
\begin{aligned}
\int_C\mathbf{v}\mathrm{d}\mathbf{x}&=\int_0^{2\pi}\begin{pmatrix}\frac{-r\sin\theta}{r^2}\\\frac{r\cos\theta}{r^2}\end{pmatrix}\cdot\begin{pmatrix}-r\sin\theta\\r\cos\theta\end{pmatrix}\mathrm{d}\theta\\
&=\int_0^{2\pi}1\mathrm{d}\theta\\
&=2\pi
\end{aligned}
$$

5　E2.1

(i)

$\cos u,\sin u$ は C^∞ から、σ も C^∞
$\sigma_u\left(u,v\right)=\begin{pmatrix}-r\sin u\\r\cos u\\0\end{pmatrix},\sigma_v\left(u,v\right)=\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$
\begin{aligned}
\sigma_u\left(u,v\right)\times\sigma_v\left(u,v\right)&=\begin{pmatrix}-r\sin u\\r\cos u\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\\
&=\begin{pmatrix}r\cos u\\r\sin u\\0\end{pmatrix}
\end{aligned}
$$

$\cos u=\sin u=0$ をみたす $u\in(0,2\pi)$ は存在しないから、$\sigma_u\left(u,v\right)\times\sigma_v\left(u,v\right)\neq\mathbf{0}$
よって $\sigma_u\left(u,v\right)$ と $\sigma_v\left(u,v\right)$ は線形独立である
$\begin{pmatrix}u\\v\end{pmatrix},\begin{pmatrix}u'\\v'\end{pmatrix}\in D,\sigma\left(u,v\right)=\sigma\left(u',v'\right)$ をする

$$
\begin{aligned}
\sigma\left(u,v\right)&=\sigma\left(u',v'\right)\\
\begin{pmatrix}r\cos u\\r\sin u\\v\end{pmatrix}&=\begin{pmatrix}r\cos u'\\r\sin u'\\v'\end{pmatrix}
\end{aligned}
$$

よって、$\cos u = \cos u', \sin u = \sin u', v = v' \Longrightarrow \tan u = \tan u', v = v' \Longrightarrow \begin{cases} u = u' \\ v = v' \end{cases}$
よって、σ は単射である
以上より、$\sigma(D)$ は σ でパラメーター表示された曲面片である

(ii)

$x^2 + y^2 = r^2 \cos^2 u + r^2 \sin^2 u = r^2 \Rightarrow \sigma(u, v) \in S_r, u \in (0, 2\pi) \Rightarrow y \neq 0 \, (\iff x \neq r)$
$\Longrightarrow \sigma(D) \subset S_r \backslash C$
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S_r \backslash C$ とすると $x^2 + y^2 = r^2 \wedge \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} r \\ 0 \end{pmatrix}$
極座標変換をすると $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos u \\ r\sin u \end{pmatrix} \wedge u \neq 2n\pi$
u の周期性を考えるとこれは $u \in (0, 2\pi)$ で $v = z$ とすれば $v \in \mathbb{R}$ も明らかに成立するから
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \sigma(D) \, (\iff S_r \backslash C \subset \sigma(D))$

6　E2.2

$\sin, \cos$ は C^∞ から σ も C^∞
$\sigma_u(u, v) = \begin{pmatrix} a\cos u \cos v \\ b\cos u \sin v \\ -c\sin u \end{pmatrix}, \sigma_v(u, v) = \begin{pmatrix} -a\sin u \sin v \\ b\sin u \cos v \\ 0 \end{pmatrix}$

$$\begin{aligned} \sigma_u(u, v) \times \sigma_v(u, v) &= \begin{pmatrix} a\cos u \cos v \\ b\cos u \sin v \\ -c\sin u \end{pmatrix} \times \begin{pmatrix} -a\sin u \sin v \\ b\sin u \cos v \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} bc\sin^2 u \cos v \\ ac\sin^2 u \sin v \\ 0 \end{pmatrix} \end{aligned}$$

$a, b, c > 0, u \in (0, \pi) \Longrightarrow \sigma_u(u, v) \times \sigma_v(u, v) \iff \sin v = \cos v = 0$
$v \in (0, 2\pi)$ から、これをみたす v は存在しない
よって $\sigma_u(u, v) \times \sigma_v(u, v) \neq \mathbf{0}$
$\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \in D, \sigma(u, v) = \sigma(u', v')$ とする

$$\sigma(u, v) = \sigma\left(u', v'\right)$$
$$\begin{pmatrix} a\sin u \cos v \\ b\sin u \sin v \\ c\cos u \end{pmatrix} = \begin{pmatrix} a\sin u' \cos v' \\ b\sin u' \sin v' \\ c\cos u' \end{pmatrix}$$

$\begin{cases} \sin u \cos v = \sin u' \cos v' \\ \sin u \sin v = \sin u' \sin v' \\ \cos u = \cos u' \end{cases} \Longrightarrow \begin{cases} \cos v = \cos v' \\ \sin v = \sin v' \\ u = u' \qquad u \in (0, \pi) \end{cases} \Longrightarrow \begin{cases} u = u' \\ v = v' \end{cases}$
よって、σ は単射である
以上より、$\sigma(D)$ は σ でパラメーター表示された曲面片である

7　P2.1

多項式だから C^∞ のは自明である
$\sigma_u = \begin{pmatrix} 1 \\ 2u+2v \\ 3u^2+6uv \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 2u \\ 3u^2 \end{pmatrix}$

$$\begin{aligned}\sigma_u \times \sigma_v &= \begin{pmatrix} 1 \\ 2u+2v \\ 3u^2+6uv \end{pmatrix} \times \begin{pmatrix} 0 \\ 2u \\ 3u^2 \end{pmatrix} \\ &= \begin{pmatrix} -6u^2v \\ -3u^2 \\ 2u \end{pmatrix}\end{aligned}$$

$u, v \neq 0 \Longrightarrow \sigma_u \times \sigma_v \neq 0$
$\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \in D$ とする.

$$\begin{pmatrix} u \\ u^2+2uv \\ u^3+3u^2v \end{pmatrix} = \begin{pmatrix} u' \\ u'^2+2u'v' \\ u'^3+3u'^2v' \end{pmatrix}$$

よって、$\begin{cases} u = u' \\ v = v' \end{cases}$

8　P2.2

e^x は C^∞ から、σ も C^∞
$\sigma_u = \begin{pmatrix} e^u - e^{-u} \\ e^u + e^{-u} \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{aligned}\sigma_u \times \sigma_v &= \begin{pmatrix} e^u - e^{-u} \\ e^u + e^{-u} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} e^u + e^{-u} \\ e^{-u} - e^u \\ 0 \end{pmatrix}\end{aligned}$$

第 1 成分は常に 0 以上であるから、$\sigma_u \times \sigma_v \neq 0$
$\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \in D$ とする.

$$\begin{pmatrix} e^u + e^{-u} \\ e^u - e^{-u} \\ v \end{pmatrix} = \begin{pmatrix} e^{u'} + e^{-u'} \\ e^{u'} - e^{-u'} \\ v' \end{pmatrix}$$

$\Longrightarrow \begin{cases} u = u' \\ v = v' \end{cases}$

9 P2.3

(i)

$\cos v, \sin v : C^\infty$ から σ も C^∞

$\sigma_u = \begin{pmatrix} a\cos v \\ b\sin v \\ 2u \end{pmatrix}, \sigma_v = \begin{pmatrix} -au\sin v \\ bu\cos v \\ 0 \end{pmatrix}$

$$\begin{aligned} \sigma_u \times \sigma_v &= \begin{pmatrix} a\cos v \\ b\sin v \\ 2u \end{pmatrix} \times \begin{pmatrix} -au\sin v \\ bu\cos v \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2bu^2\cos v \\ -2au^2\sin v \\ abu \end{pmatrix} \end{aligned}$$

$u \neq 0, abu \neq 0, \sigma_u \times \sigma_v \neq 0$

$\begin{pmatrix} au\cos v \\ bu\sin v \\ u^2 \end{pmatrix} = \begin{pmatrix} au'\cos v' \\ bu'\sin v' \\ u'^2 \end{pmatrix} \Longrightarrow \begin{cases} u = u' \\ \cos v = \cos v' \\ \sin v = \sin v' \end{cases} \Longrightarrow \begin{cases} u = u' \\ v = v' \end{cases}$

(ii)

$0 \notin (0, 2\pi) \Longrightarrow \sin v \neq 0 \Longrightarrow y \neq 0$

$a, b, u > 0$ から、$\begin{pmatrix} au\cos v \\ bu\sin v \end{pmatrix} \notin (0, \infty) \times \{0\}$

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{(au\cos v)^2}{a^2} + \dfrac{(bu\sin v)^2}{b^2} = u^2\cos^2 v + u^2\sin^2 v = u^2 = z$

$\therefore \sigma(D) \subset S \backslash C$

$\sigma(u, v) \in S \backslash C$ とする

$\begin{pmatrix} au\cos v \\ bu\sin v \\ u^2 \end{pmatrix} \in \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \right\}$ は常に成立する

$\sigma(u, v) \notin C \Longrightarrow bu\sin v \neq 0 \Longrightarrow v \neq 2n\pi$

$\Longrightarrow \sigma(u, v) \in \sigma(D) \Longrightarrow \sigma(D) \supset S \backslash C$

$\therefore \sigma(D) = S \backslash C$

10 3.1

(1)

$\sigma_u = \begin{pmatrix} -r\sin u \\ r\cos u \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \sigma_u \times \sigma_v = \begin{pmatrix} r\cos u \\ r\sin u \\ 0 \end{pmatrix}$

$p\left(\dfrac{\pi}{3}, 1\right)$ を代入すると $\sigma_u \times \sigma_v = \begin{pmatrix} \frac{1}{2}r \\ \frac{\sqrt{3}}{2}r \\ 0 \end{pmatrix}$

よって、接平面の方程式は $\begin{pmatrix} x - \frac{1}{2}r \\ y - \frac{\sqrt{3}}{2}r \\ z - 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}r \\ \frac{\sqrt{3}}{2}r \\ 0 \end{pmatrix} = 0$

(2)

$$\begin{aligned}\mathbf{n} &= \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} \\ &= \frac{1}{r}\begin{pmatrix} r\cos u \\ r\sin u \\ 0 \end{pmatrix}\end{aligned}$$

11　3.2

(1)

多項式は C^∞ から、σ も C^∞
$\sigma_u = \begin{pmatrix} 2u \\ 3u^2-2 \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \sigma_u \times \sigma_v = \begin{pmatrix} 3u^2-2 \\ -2u \\ 0 \end{pmatrix}$
$3u^2-2=-2u=0$ をみたす u は存在しないから、$\sigma_u \times \sigma_v \neq 0$
$(u,v), (u',v') \in D, \sigma(u,v) = \sigma(u',v')$ とする
$\Longrightarrow \begin{pmatrix} r\cos u \\ r\sin u \\ v \end{pmatrix} = \begin{pmatrix} r\cos u' \\ r\sin u' \\ v' \end{pmatrix} \Longrightarrow \begin{cases} v = v' \\ \cos u = \cos u' \\ \sin u = \sin u' \end{cases} \Longrightarrow \begin{cases} u = u' \\ v = v' \end{cases}$

(2)

p を代入すると、$\sigma_u \times \sigma_v = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$
よって、接平面の方程式は $\begin{pmatrix} x+2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} = 0$

(3)

$$\begin{aligned}\mathbf{n} &= \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} \\ &= \frac{1}{\sqrt{9u^4-8u^2+4}}\begin{pmatrix} 3u^2-2 \\ -2u \\ 0 \end{pmatrix}\end{aligned}$$

12　3.1

(1)

$\sigma_u = \begin{pmatrix} 1 \\ 0 \\ u \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 1 \\ v \end{pmatrix}, \sigma_u \times \sigma_v = \begin{pmatrix} -u \\ -v \\ 1 \end{pmatrix}$
$\Pi : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \iff z = 0$

(2)

$\|\sigma_u \times \sigma_v\| = \sqrt{u^2 + v^2 + 1}$

$\mathbf{n} = \dfrac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \dfrac{1}{\sqrt{u^2 + v^2 + 1}} \begin{pmatrix} -u \\ -v \\ 1 \end{pmatrix}$

13 3.2

(1)

$\sigma_u = \begin{pmatrix} 6 \\ 0 \\ -2 \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 6 \\ -3 \end{pmatrix}, \sigma_u \times \sigma_v = \begin{pmatrix} 12 \\ 18 \\ 36 \end{pmatrix}$

$\Pi : \begin{pmatrix} x-6 \\ y-6 \\ z+4 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 18 \\ 36 \end{pmatrix} = 0 \iff 2(x-6) + 3(y-6) + 6(z+4) = 0$

(2)

$\|\sigma_u \times \sigma_v\| = \sqrt{12^2 + 18^2 + 36^2} = 42$

$$\begin{aligned} \mathbf{n} &= \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} \\ &= \frac{1}{42} \begin{pmatrix} 12 \\ 18 \\ 36 \end{pmatrix} \\ &= \frac{1}{7} \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \end{aligned}$$

14 3.3

(1)

$\sigma_u = \begin{pmatrix} 1 \\ 2u + 2v \\ 3u^2 + 6uv \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 2u \\ 3u^2 \end{pmatrix}, \sigma_u \times \sigma_v = \begin{pmatrix} -6u^2 v \\ -3u^2 \\ 2u \end{pmatrix}$

$p = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} u \\ u^2 + 2uv \\ u^3 + 3u^2 v \end{pmatrix} \Longrightarrow \begin{cases} u = 1 \\ v = -1 \end{cases}$

$\Pi : \begin{pmatrix} x-1 \\ y+1 \\ z+2 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} = 0 \iff 6(x-1) - 3(y+1) + 2(z+2) = 0$

(2)

$\|\sigma_u \times \sigma_v\| = \sqrt{36u^4v^2 + 9u^4 + 4u^2} = u\sqrt{9u^2\left(4v^2+1\right)+4}$

$$
\begin{aligned}
\mathbf{n} &= \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} \\
&= \frac{1}{u\sqrt{9u^2\left(4v^2+1\right)+4}}\begin{pmatrix} -6u^2v \\ -3u^2 \\ 2u \end{pmatrix} \\
&= \frac{1}{\sqrt{9u^2\left(4v^2+1\right)+4}}\begin{pmatrix} -6uv \\ -3u \\ 2 \end{pmatrix}
\end{aligned}
$$

15　4.1

$\sigma = \begin{pmatrix} u \\ v \\ \sqrt{r^2-u^2-v^2} \end{pmatrix}, \sigma_u = \begin{pmatrix} 1 \\ 0 \\ -\frac{u}{\sqrt{r^2-u^2-v^2}} \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 1 \\ -\frac{v}{\sqrt{r^2-u^2-v^2}} \end{pmatrix}$

$\sigma_u \times \sigma_v = \begin{pmatrix} \frac{u}{\sqrt{r^2-u^2-v^2}} \\ \frac{v}{\sqrt{r^2-u^2-v^2}} \\ 1 \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \frac{r}{\sqrt{r^2-u^2-v^2}}$

(1)

$$
\begin{aligned}
Area\left(T\right) &= \iint_{\Omega} \|\sigma_u \times \sigma_v\| \,\mathrm{d}u\mathrm{d}v \\
&= \iint_{u^2+v^2<a^2} \frac{r}{\sqrt{r^2-u^2-v^2}}\mathrm{d}u\mathrm{d}v \\
&= \int_0^{2\pi}\int_0^{a} \frac{r}{\sqrt{r^2-\delta^2}} \cdot \delta\mathrm{d}\delta\mathrm{d}\theta \\
&= \int_0^{2\pi}\left(r^2 - r\sqrt{r^2-\sqrt{a}}\right)\mathrm{d}\theta \\
&= 2\pi r\left(r - \sqrt{r^2-\sqrt{a}}\right)
\end{aligned}
$$

(2)

$$
\begin{aligned}
\iint_T f\mathrm{d}A &= \iint_\Omega \left(4u^2v^2\sqrt{r^2-u^2-v^2}\right)\cdot\frac{r}{\sqrt{r^2-u^2-v^2}}\mathrm{d}u\mathrm{d}v \\
&= 4r\iint_{u^2+v^2<a^2} u^2v^2\mathrm{d}u\mathrm{d}v \\
&= 4r\int_0^{2\pi}\int_0^a \delta^4\sin^2\theta\cos^2\theta\cdot\delta\mathrm{d}\delta\mathrm{d}\theta \\
&= 4r\int_0^{2\pi}\sin^2\theta\cos^2\theta\left(\int_0^a\delta^5\mathrm{d}\delta\right)\mathrm{d}\theta \\
&= \frac{2}{3}ra^6\int_0^{2\pi}\sin^2\theta\cos^2\theta\mathrm{d}\theta \\
&= \frac{1}{6}ra^6\int_0^{2\pi}\sin^2 2\theta\mathrm{d}\theta \\
&= \frac{1}{12}ra^6\int_0^{2\pi}(1-\cos 4\theta)\,\mathrm{d}\theta \\
&= \frac{1}{6}ra^6\pi
\end{aligned}
$$

(3)

$$
\begin{aligned}
\iint_T \mathbf{v}_1\cdot\mathrm{d}\mathbf{A} &= \iint_\Omega \begin{pmatrix}0\\0\\3\end{pmatrix}\cdot\begin{pmatrix}\frac{u}{\sqrt{r^2-u^2-v^2}}\\ \frac{v}{\sqrt{r^2-u^2-v^2}}\\ 1\end{pmatrix}\mathrm{d}u\mathrm{d}v \\
&= 3\int_0^{2\pi}\int_0^a\delta\mathrm{d}\delta\mathrm{d}\theta \\
&= 3\int_0^{2\pi}\frac{1}{2}a^2\mathrm{d}\theta \\
&= 3\pi a^2
\end{aligned}
$$

(4)

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a\cos t \\ a\sin t \end{pmatrix}, C\left(t\right) = \sigma\left(u, v\right) = \begin{pmatrix} a\cos t \\ a\sin t \\ \sqrt{r^2 + a^2} \end{pmatrix}$$

$$\mathbf{v}_2\left(C\left(t\right)\right) = \begin{pmatrix} a\cos t - 3a\sin t \\ a\sin t\left(r^2 - a^2\right) \\ a^2\sin^2 t\sqrt{r^2 - a^2} \end{pmatrix}, C'\left(t\right) = \begin{pmatrix} -a\sin t \\ a\cos t \\ 0 \end{pmatrix}$$

$$\begin{aligned} \iint_{\sigma\left(\overline{\Omega}\right)} \nabla \times \mathbf{v}_2 \cdot \mathrm{d}\mathbf{A} &= \oint_{\partial T} \mathbf{v}_2 \cdot \mathrm{d}\mathbf{r} \\ &= \int_0^{2\pi} \left(a^2\left(r^2 - a^2 - a\right)\sin t\cos t + 3a^2\sin^2 t\right)\mathrm{d}t \\ &= a^2\left(r^2 - a^2 - 1\right)\int_0^{2\pi} \sin t\cos t\mathrm{d}t + 3a^2\int_0^{2\pi} \sin^2 t\mathrm{d}t \\ &= a^2\left(r^2 - a^2 - 1\right)\left[-\frac{1}{4}\cos 2t\right]_0^{2\pi} + 3a^2\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{2\pi} \\ &= 3\pi a^2 \end{aligned}$$

(5)

$$\mathbf{v}_3\left(\sigma\left(u, v\right)\right) = \begin{pmatrix} u\sqrt{r^2 - u^2 - v^2} \\ v\sqrt{r^2 - u^2 - v^2} \\ r^2 - \left(r^2 - u^2 - v^2\right) \end{pmatrix} = \begin{pmatrix} u\sqrt{r^2 - u^2 - v^2} \\ v\sqrt{r^2 - u^2 - v^2} \\ u^2 + v^2 \end{pmatrix}, \sigma_u \times \sigma_v = \begin{pmatrix} \frac{u}{\sqrt{r^2 - u^2 - v^2}} \\ \frac{v}{\sqrt{r^2 - u^2 - v^2}} \\ 1 \end{pmatrix}$$

$$\begin{aligned} \iint_T \mathbf{v}_3 \cdot \mathrm{d}\mathbf{A} &= \iint_\Omega \begin{pmatrix} u\sqrt{r^2 - u^2 - v^2} \\ v\sqrt{r^2 - u^2 - v^2} \\ u^2 + v^2 \end{pmatrix} \cdot \begin{pmatrix} \frac{u}{\sqrt{r^2 - u^2 - v^2}} \\ \frac{v}{\sqrt{r^2 - u^2 - v^2}} \\ 1 \end{pmatrix} \mathrm{d}u\mathrm{d}v \\ &= 2\iint_\Omega \left(u^2 + v^2\right)\mathrm{d}u\mathrm{d}v \\ &= 2\int_0^{2\pi}\int_0^a \delta^2 \cdot \delta\mathrm{d}\delta\mathrm{d}t \\ &= \frac{1}{2}a^4\int_0^{2\pi}\mathrm{d}t \\ &= a^4\pi \end{aligned}$$

(6)

$$C\left(t\right)=\sigma\left(u,v\right)=\begin{pmatrix} a\cos t\\ a\sin t\\ \sqrt{r^2-a^2}\end{pmatrix},C'\left(t\right)=\begin{pmatrix} -a\sin t\\ a\cos t\\ 0\end{pmatrix}$$

$$\mathbf{v}_4\left(C\left(t\right)\right)=\begin{pmatrix} a\sin t\cdot\left(r^2-a^2\right)\\ r^2\cdot a\cos t\\ a^2\sin t\cos t\cdot\sqrt{r^2-a^2}\end{pmatrix}$$

$$\begin{aligned}\iint_{\sigma(\overline{\Omega})}\nabla\times\mathbf{v}_4\cdot\mathrm{d}\mathbf{A}&=\oint_{\partial T}\mathbf{v}_4\cdot\mathrm{d}r\\&=\oint_{\partial T}\begin{pmatrix} a\sin t\cdot\left(r^2-a^2\right)\\ r^2\cdot a\cos t\\ a^2\sin t\cos t\cdot\sqrt{r^2-a^2}\end{pmatrix}\cdot\begin{pmatrix} -a\sin t\\ a\cos t\\ 0\end{pmatrix}\mathrm{d}t\\&=\int_0^{2\pi}\left(-a^2\left(r^2-a^2\right)\sin^2 t+a^2r^2\cos^2 t\right)\mathrm{d}t\\&=\left(a^4-a^2r^2\right)\int_0^{2\pi}\sin^2 t\mathrm{d}t+a^2r^2\int_0^{2\pi}\cos^2 t\mathrm{d}t\\&=\frac{1}{2}\left(a^4-a^2r^2\right)\left[t-\frac{1}{2}\sin 2t\right]_0^{2\pi}+\frac{1}{2}a^2r^2\left[t+\frac{1}{2}\sin 2t\right]_0^{2\pi}\\&=\pi a^4\end{aligned}$$

16　4.1

$$\sigma_u=\begin{pmatrix}1\\0\\2u\end{pmatrix},\sigma_v=\begin{pmatrix}0\\1\\-2v\end{pmatrix},\sigma_u\times\sigma_v=\begin{pmatrix}-2u\\2v\\1\end{pmatrix},\left\|\sigma_u\times\sigma_v\right\|=\sqrt{4u^2+4v^2+1}$$

(1)

$$\begin{aligned}Area\left(T\right)&=\iint_{\Omega}\left\|\sigma_u\times\sigma_v\right\|\mathrm{d}u\mathrm{d}v\\&=\iint_{\Omega}\sqrt{4u^2+4v^2+1}\mathrm{d}u\mathrm{d}v\\&=\int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\sqrt{4\delta^2+1}\cdot\delta\mathrm{d}\delta\mathrm{d}\theta\\&=\int_0^{2\pi}\int_0^{\frac{3}{4}}\frac{1}{2}\sqrt{4t+1}\mathrm{d}t\mathrm{d}\theta\\&=\frac{1}{12}\int_0^{2\pi}\left[(4t+1)^{\frac{3}{2}}\right]_0^{\frac{3}{4}}\mathrm{d}\theta\\&=\frac{1}{12}\int_0^{2\pi}7\mathrm{d}\theta\\&=\frac{7}{6}\pi\end{aligned}$$

(2)

$$
\begin{aligned}
\iint_T f \mathrm{d}A &= \iint_\Omega \left(u^2+v^2\right)\sqrt{4u^2+4v^2+1}\mathrm{d}u\mathrm{d}v \\
&= \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}} \delta^2 \cdot \sqrt{4\delta^2+1}\cdot \delta \mathrm{d}\delta \mathrm{d}\theta \\
&= \frac{1}{32}\int_0^{2\pi}\int_1^4 \left(t^{\frac{3}{2}} - t^{\frac{1}{2}}\right)\mathrm{d}t\mathrm{d}\theta \\
&= \frac{1}{32}\int_0^{2\pi}\left[\frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}}\right]_1^4 \mathrm{d}\theta \\
&= \frac{1}{32}\int_0^{2\pi}\frac{116}{25}\mathrm{d}\theta \\
&= \frac{29}{60}\pi
\end{aligned}
$$

(3)

$$
\begin{aligned}
\iint_T \mathbf{v}_1 \cdot \mathrm{d}\mathbf{A} &= \iint_\Omega \begin{pmatrix} -u\left(u^2-v^2\right) \\ v\left(u^2-v^2\right) \\ u^2+v^2 \end{pmatrix} \cdot \begin{pmatrix} -2u \\ 2v \\ 1 \end{pmatrix} \mathrm{d}u\mathrm{d}v \\
&= \iint_\Omega \left(u^2+v^2\right)\left(2u^2-2v^2+1\right)\mathrm{d}u\mathrm{d}v \\
&= \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}} k^2\left(2k^2\cos 2\theta + 1\right)\cdot k\mathrm{d}k\mathrm{d}\theta \\
&= \int_0^{2\pi}\left(\cos 2\theta\left[\frac{1}{3}k^6\right]_0^{\frac{\sqrt{3}}{2}} + \left[\frac{1}{4}k^4\right]_0^{\frac{\sqrt{3}}{2}}\right)\mathrm{d}\theta \\
&= \int_0^{2\pi}\left(\frac{9}{64}\cos 2\theta + \frac{9}{64}\right)\mathrm{d}\theta \\
&= \frac{9}{64}\int_0^{2\pi}(\cos 2\theta + 1)\mathrm{d}\theta \\
&= \frac{9}{64}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} \\
&= \frac{9}{32}\pi
\end{aligned}
$$

(4)

$$C(t) = \sigma(u,v) = \begin{pmatrix} a\cos t \\ a\sin t \\ a^2\cos^2 t - a^2\sin^2 t \end{pmatrix} = \begin{pmatrix} a\cos t \\ a\sin t \\ a^2\cos 2t \end{pmatrix}, C'(t) = \begin{pmatrix} -a\sin t \\ a\cos t \\ -2a^2\sin 2t \end{pmatrix}$$

$$\begin{aligned}\iint_{\sigma(\overline{\Omega})} \nabla\times\mathbf{v}_2\cdot \mathrm{d}\mathbf{A} &= \oint_{\partial T} \begin{pmatrix} -a\sin t \\ a\cos t \\ a^2\cos 2t \end{pmatrix}\cdot\begin{pmatrix} -a\sin t \\ a\cos t \\ -2a^2\sin 2t \end{pmatrix}\mathrm{d}t \\ &= a^2\int_0^{2\pi}\left(1 - a^2\sin 4t\right)\mathrm{d}t \\ &= a^2\left[t + \frac{1}{4}a^2\cos 4t\right]_0^{2\pi} \\ &= 2a^2\pi\end{aligned}$$

17　4.2

$$\sigma(u,v) = \begin{pmatrix} r\sin u\cos v \\ r\sin u\sin v \\ r\cos u \end{pmatrix}, \sigma_u = \begin{pmatrix} r\cos u\cos v \\ r\cos u\sin v \\ -r\sin u \end{pmatrix}, \sigma_v = \begin{pmatrix} -r\sin u\sin v \\ r\sin u\cos v \\ 0 \end{pmatrix}$$

$$\sigma_u\times\sigma_v = \begin{pmatrix} r^2\sin^2 u\cos v \\ r^2\sin^2 u\sin v \\ r^2\sin u\cos u \end{pmatrix}, \|\sigma_u\times\sigma_v\| = r^2\sin u$$

(1)

$$\begin{aligned} Area(T) &= \iint_\Omega \|\sigma_u\times\sigma_v\|\,\mathrm{d}u\mathrm{d}v \\ &= \iint_\Omega r^2\sin u\mathrm{d}u\mathrm{d}v \\ &= r^2\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\int_{\frac{\pi}{3}}^{\frac{2}{3}\pi}\sin u\mathrm{d}u\mathrm{d}v \\ &= r^2\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\left[-\cos u\right]_{\frac{\pi}{3}}^{\frac{2}{3}\pi}\mathrm{d}v \\ &= r^2\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\mathrm{d}v \\ &= \pi r^2\end{aligned}$$

(2)

$$\begin{aligned}\iint_S f\mathrm{d}A &= \iint_D \left(r\sin u\cos v\cdot r\sin u\sin v\cdot r^2\cos^2 u\right)\cdot\left(r^2\sin u\right)\mathrm{d}u\mathrm{d}v\\ &= \iint_D \left(r^6\sin^3 u\cos^2 u\sin v\cos v\right)\mathrm{d}u\mathrm{d}v\\ &= \int_0^{2\pi}\int_0^{\pi} r^6\sin^3 u\cos^2 u\sin v\cos v\mathrm{d}u\mathrm{d}v\\ &= r^6\int_0^{2\pi}\sin v\cos v\left(\int_0^{\pi}\sin^3 u\cos^2 u\mathrm{d}u\right)\mathrm{d}v\\ &= \frac{4}{15}r^6\int_0^{2\pi}\sin v\cos v\mathrm{d}v\\ &= 0\end{aligned}$$

(3)

$$\begin{aligned}\iint_S \mathbf{v}_1\cdot\mathrm{d}\mathbf{A} &= \iint_D \begin{pmatrix}0\\0\\r^3\sin^3 u\cos^3 v\end{pmatrix}\cdot\begin{pmatrix}r^2\sin^2 u\cos v\\ r^2\sin^2 u\sin v\\ r^2\sin u\cos u\end{pmatrix}\mathrm{d}u\mathrm{d}v\\ &= \iint_D r^5\sin^4 u\cos u\cos^3 v\mathrm{d}u\mathrm{d}v\\ &= r^5\int_0^{2\pi}\cos^3 v\left(\int_0^{\pi}\sin^4 u\cos u\mathrm{d}u\right)\mathrm{d}v\\ &= r^5\int_0^{2\pi}0\cdot\cos^3 v\mathrm{d}v\\ &= 0\end{aligned}$$

(4)

$\mathbf{v}_2 = \begin{pmatrix}yz^2\\xz^2\\2xyz\end{pmatrix}, \nabla\times\mathbf{v}_2 = \begin{pmatrix}2xz-2xz\\2yz-2yz\\z^2-z^2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ から

$$\iint_{\sigma(\overline{\Omega})}\nabla\times\mathbf{v}_2\cdot\mathrm{d}\mathbf{A} = \oint_{\partial\Omega}0\cdot\mathrm{d}r = 0$$

18　4.3

$\sigma(u,v) = \begin{pmatrix}u\\v\\u^2+v^2\end{pmatrix}, \sigma_u = \begin{pmatrix}1\\0\\2u\end{pmatrix}, \sigma_v = \begin{pmatrix}0\\1\\2v\end{pmatrix}$

$\sigma_u\times\sigma_v = \begin{pmatrix}-2u\\-2v\\1\end{pmatrix}, \|\sigma_u\times\sigma_v\| = \sqrt{4u^2+4v^2+1}$

(1)

$$
\begin{aligned}
Area\left(T\right) &= \iint_{\Omega} \left\|\sigma_u \times \sigma_v\right\| \mathrm{d}u\mathrm{d}v \\
&= \int_0^{2\pi} \int_2^9 \sqrt{4\delta^2+1}\delta \cdot \mathrm{d}\delta\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} \left(1625\sqrt{13} - 17\sqrt{17}\right) \mathrm{d}t \\
&= \frac{1}{6}\left(1625\sqrt{13} - 17\sqrt{17}\right)\pi
\end{aligned}
$$

(2)

$$
C_1\left(t\right) = \begin{pmatrix} 9\cos t \\ 9\sin t \\ 81 \end{pmatrix}, C_2\left(t\right) = \begin{pmatrix} 2\cos t \\ -2\sin t \\ 4 \end{pmatrix}, C_1'\left(t\right) = \begin{pmatrix} -9\sin t \\ 9\cos t \\ 0 \end{pmatrix}, C_2'\left(t\right) = \begin{pmatrix} -2\sin t \\ -2\cos t \\ 0 \end{pmatrix}
$$

$$
\mathbf{v}_1\left(C_1\right) = \begin{pmatrix} 81\cos^2 t + 9\sin t - 4 \\ 243\sin t\cos t \\ 1458\cos t + 6561 \end{pmatrix}, \mathbf{v}_1\left(C_2\right) = \begin{pmatrix} 8 - 4\cos^2 t \\ 8\sin t\cos t + 2\sin t \\ -4\sin t \end{pmatrix}
$$

$$
\begin{aligned}
\iint_{\sigma\left(\overline{\Omega}\right)} \nabla \times \mathbf{v}_1 \cdot \mathrm{d}\mathbf{A} &= \int_0^{2\pi} \left(1450\sin t\cos^2 t - 81\sin^2 t - 4\sin t\cos t + 20\sin t\right) \mathrm{d}t \\
&= -81\pi
\end{aligned}
$$

(3)

$$
C_1\left(t\right) = \begin{pmatrix} 9\cos t \\ 9\sin t \\ 81 \end{pmatrix}, C_2\left(t\right) = \begin{pmatrix} 2\cos t \\ -2\sin t \\ 4 \end{pmatrix}, C_1'\left(t\right) = \begin{pmatrix} -9\sin t \\ 9\cos t \\ 0 \end{pmatrix}, C_2'\left(t\right) = \begin{pmatrix} -2\sin t \\ -2\cos t \\ 0 \end{pmatrix}
$$

$$
\mathbf{v}_2\left(C_1\right) = \begin{pmatrix} 162 - 81\cos^2 t \\ -162\sin t\cos t - 9\sin t \\ 18\sin t \end{pmatrix}, \mathbf{v}_2\left(C_2\right) = \begin{pmatrix} 8 - 4\cos^2 t \\ 8\sin t\cos t + 2\sin t \\ -4\sin t \end{pmatrix}
$$

$$
\begin{aligned}
\iint_{\sigma\left(\overline{\Omega}\right)} \nabla \times \mathbf{v}_2 \cdot \mathrm{d}\mathbf{A} &= \int_0^{2\pi} \left(478\sin t\cos^2 t + 14\sin t\cos t - 340\sin t\right) \mathrm{d}t \\
&= 0
\end{aligned}
$$

19　5.1

(1)

$f \overset{def}{:=} x\sin\dfrac{z}{k} - y\cos\dfrac{z}{k} = 0$ とする、$\forall \mathbf{p} \in S, \forall \epsilon > 0, \forall U = N\left(\mathbf{p}, \epsilon\right) \cap S = f^{-1}\left(\{0\}\right)$

また、$\nabla f = \begin{pmatrix} \sin\frac{z}{k} \\ -\cos\frac{z}{k} \\ \frac{x}{k}\cos\frac{z}{k} + \frac{y}{k}\sin\frac{z}{k} \end{pmatrix} = 0 \Longrightarrow \begin{cases} \sin\frac{z}{k} = 0 \\ -\cos\frac{z}{k} = 0 \\ \frac{x}{k}\cos\frac{z}{k} + \frac{y}{k}\sin\frac{z}{k} = 0 \end{cases}$

$\sin\frac{z}{k} = -\cos\frac{z}{k} = 0$ をみたす $z \in \mathbb{R}$ は存在しないから、$\left(\nabla f\right)\left(\mathbf{p}\right) \neq 0$

以上より、S は正則曲面である

(2)

$\cos v, \sin v : C^\infty$ から、$\sigma : C^\infty$
$\sigma_u = \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} -u\sin v \\ u\cos v \\ k \end{pmatrix}$

$$\begin{aligned} \sigma_u \times \sigma_v &= \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix} \times \begin{pmatrix} -u\sin v \\ u\cos v \\ k \end{pmatrix} \\ &= \begin{pmatrix} k\sin v \\ -k\cos v \\ u \end{pmatrix} \end{aligned}$$

$\sin v = \cos v = 0$ をみたす v は存在しないから、$\sigma_u \times \sigma_v \neq 0$ で、σ_u と σ_v は線形独立

$$\begin{pmatrix} u\cos v \\ u\sin v \\ kv \end{pmatrix} = \begin{pmatrix} u'\cos v' \\ u'\sin v' \\ kv' \end{pmatrix}$$

とすると、第三成分から $v = v'$ で、第一成分または第二成分に代入すると、$u = u'$
よって、σ は単射である. 以上より、σ は局所パラメーター表示である

(3)

$S \subset \sigma\left(\mathbb{R}^2\right)$

$\mathbf{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S$ とすると $x\sin\dfrac{z}{k} - y\cos\dfrac{z}{k} = 0$
このとき、$v = \dfrac{z}{k}, u = x\cos v + y\sin v$ とすると

$$\begin{aligned} \sigma(u,v) &= \begin{pmatrix} u\cos v \\ u\sin v \\ kv \end{pmatrix} \\ &= \begin{pmatrix} \left(x\cos\frac{z}{k} + y\sin\frac{z}{k}\right)\cos\frac{z}{k} \\ \left(x\cos\frac{z}{k} + y\sin\frac{z}{k}\right)\sin\frac{z}{k} \\ k\frac{z}{k} \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \sigma\left(\mathbb{R}^2\right) \end{aligned}$$

$S \supset \sigma\left(\mathbb{R}^2\right)$

$$\sigma(u,v) = \begin{pmatrix} u\cos v \\ u\sin v \\ kv \end{pmatrix}$$

とする

$$\begin{aligned} x\sin\frac{z}{k} - y\cos\frac{z}{k} &= (u\cos v)\sin v - (u\sin v)\cos v \\ &= 0 \end{aligned}$$

から、$\sigma\left(\mathbb{R}^2\right) \subset S$
　以上より $S = \sigma\left(\mathbb{R}^2\right)$

20　5.2

(1)

$f = x^2 - y^2 - r^2$ とすると $\forall \mathbf{p} \in S, \forall \epsilon > 0, \forall U = N(\mathbf{p}, \epsilon) \cap S = f^{-1}(\{0\})$
$\nabla f = \begin{pmatrix} 2x \\ -2y \\ 0 \end{pmatrix}$ で、$\forall \mathbf{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S, x^2 = y^2 + r^2 > 0$ から、$(\nabla f)(\mathbf{p}) \neq 0$
以上より、S が正則曲面である

(2)

σ_+ と σ_- は第一成分の符号だけ違っているから、σ_+ だけ確認すればいい
$\cosh u, \sinh u, v : C^\infty$ から $\sigma_+ : C^\infty$
$\sigma_{+_u} = \begin{pmatrix} r\sinh u \\ r\cosh u \\ 0 \end{pmatrix}, \sigma_{+_v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{aligned} \sigma_{+_u} \times \sigma_{+_v} &= \begin{pmatrix} r\sinh u \\ r\cosh u \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} r\cosh u \\ -r\sinh u \\ 0 \end{pmatrix} \end{aligned}$$

$r\cosh u = r\sinh u = 0$ をみたす u は存在しないから、$\sigma_{+_u} \times \sigma_{+_v} \neq 0$ で、σ_{+_u} と σ_{+_v} は線形独立

$$\begin{pmatrix} r\cosh u \\ r\sinh u \\ v \end{pmatrix} = \begin{pmatrix} r\cosh u' \\ r\sinh u' \\ v' \end{pmatrix}$$

よって $u = u', v = v'$ だから、σ_+ は単射である

(3)

$S \subset \sigma_+\left(\mathbb{R}^2\right) \cup \sigma_-\left(\mathbb{R}^2\right)$

$\mathbf{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S$ とすると $x^2 - y^2 = r^2$
$x^2 \mapsto (r\cosh u)^2, y \mapsto r\sinh u, z \mapsto v$ をみたす全単射が存在するから、$\mathbf{p} \in \sigma_+\left(\mathbb{R}^2\right) \cup \sigma_-\left(\mathbb{R}^2\right)$

$S \supset \sigma_+\left(\mathbb{R}^2\right) \cup \sigma_-\left(\mathbb{R}^2\right)$

$\mathbf{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \int \sigma_+\left(\mathbb{R}^2\right) \cup \sigma_-\left(\mathbb{R}^2\right)$ とする

$(r\cosh u)^2 - (r\sinh u)^2 = (-r\cosh u)^2 - (r\sinh u)^2 = r^2\left(\cosh^2 u - \sinh^2 u\right) = r^2$

よって、$\mathbf{p} \in S$

21　5.1

(1)

$f := x^2 + y^2 - a^2\cosh^2\frac{z}{a}$ とすると、$\forall \mathbf{p} \in S, \forall \epsilon > 0, \forall U = N(\mathbf{p}, \epsilon) \cap S = f^{-1}(\{0\})$

$\nabla f = \begin{pmatrix} 2x \\ 2y \\ -2a\sinh\frac{z}{a}\cosh\frac{z}{a} \end{pmatrix}$ で、$(\nabla f)(\mathbf{p}) = 0$ となる $\mathbf{p}$ は $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ であり、$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin S$ だから $(\nabla f)(\mathbf{p}) \neq 0$

(2)

$\cosh u, \cos v, \sin v : C^\infty$ から $\sigma : C^\infty$

$\sigma_u = \begin{pmatrix} a\sinh u\cos v \\ a\sinh u\sin v \\ a \end{pmatrix}, \sigma_v = \begin{pmatrix} -a\cosh u\sin v \\ a\cosh u\cos v \\ 0 \end{pmatrix}$

$$\begin{aligned} \sigma_u \times \sigma_v &= \begin{pmatrix} a\sinh u\cos v \\ a\sinh u\sin v \\ a \end{pmatrix} \times \begin{pmatrix} -a\cosh u\sin v \\ a\cosh u\cos v \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -a^2\cosh u\cos v \\ -a^2\cosh u\sin v \\ a^2\sinh u\cosh u \end{pmatrix} \end{aligned}$$

$\cosh u > 0$ かつ $\cos v = \sin v = 0$ をみたす $v \in (0, 2\pi)$ は存在しないから、$\sigma_u \times \sigma_v \neq 0$

よって、σ_u と σ_v は線形独立

$$\begin{pmatrix} a\cosh u\cos v \\ a\cosh u\sin v \\ au \end{pmatrix} = \begin{pmatrix} a\cosh u'\cos v' \\ a\cosh u'\sin v' \\ au' \end{pmatrix}$$

とすると、$u = u'$ で、$\begin{cases} \cos v = \cos v' \\ \sin v = \sin v' \end{cases}, v = v'$

よって、σ は単射である

22　5.2

(1)

$f = x^2 + y^2 - z^2 - r^2$ とすると、$\forall \mathbf{p} \in S, \forall \epsilon > 0, \forall U = N(\mathbf{p}, \epsilon) \cap S = f^{-1}(\{0\})$

$\nabla f = \begin{pmatrix} 2x \\ 2y \\ -2z \end{pmatrix}$ で、$(\nabla f)(\mathbf{p}) = 0$ をみたす $\mathbf{p}$ は $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ で S に属しないから、$(\nabla f)(p) \neq 0$

(2)

$\cosh u, \sinh u, \cos v, \sin v : C^\infty$ から $\sigma : C^\infty$

$\sigma_u = \begin{pmatrix} r\sinh u\cos v \\ r\sinh u\sin v \\ r\cosh u \end{pmatrix}, \sigma_v = \begin{pmatrix} -r\cosh u\sin v \\ r\cosh u\cos v \\ 0 \end{pmatrix}$

$$\begin{aligned}\sigma_u \times \sigma_v &= \begin{pmatrix} r\sinh u\cos v \\ r\sinh u\sin v \\ r\cosh u \end{pmatrix} \times \begin{pmatrix} -r\cosh u\sin v \\ r\cosh u\cos v \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -r^2\cosh^2 u\cos v \\ -r^2\cosh^2 u\sin v \\ r^2\sinh u\cosh u \end{pmatrix}\end{aligned}$$

$\cosh u > 0$ かつ $\cos v = \sin v = 0$ をみたす $v \in (0, 2\pi)$ は存在しないから、$\sigma_u \times \sigma_v \neq 0$
よって、σ_u と σ_v は線形独立

$$\begin{pmatrix} r\cosh u\cos v \\ r\cosh u\sin v \\ r\sinh u \end{pmatrix} = \begin{pmatrix} r\cosh u'\cos v' \\ r\cosh u'\sin v' \\ r\sinh u \end{pmatrix}$$

第三成分より $u = u', \begin{cases} \cos v = \cos v' \\ \sin v = \sin v' \end{cases}, v = v'$
よって、σ は単射である

23　5.3

(1)

$f = x^2 + y^2 - z^2 + r^2$ とすると、$\forall \mathbf{p} \in S, \forall \epsilon > 0, \forall U = N(\mathbf{p}, \epsilon) \cap S = f^{-1}(\{0\})$
$\nabla f = \begin{pmatrix} 2x \\ 2y \\ -2z \end{pmatrix} = 0$ をみたす $\mathbf{p}$ は $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ で $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin S$ から、$(\nabla f)(\mathbf{p}) \neq 0$

(2)

σ_+ と σ_- は第三成分の符号の違いだけあるから、σ_+ だけ考えればいい
$u, v, \sqrt{r^2 + u^2 + v^2} : C^\infty$ から $\sigma_+ : C^\infty$

$\sigma_{+_u} = \begin{pmatrix} 1 \\ 0 \\ \dfrac{u}{\sqrt{r^2+u^2+v^2}} \end{pmatrix}, \sigma_{+_v} = \begin{pmatrix} 0 \\ 1 \\ \dfrac{v}{\sqrt{r^2+u^2+v^2}} \end{pmatrix}$

$$\begin{aligned}\sigma_{+_u} \times \sigma_{+_v} &= \begin{pmatrix} 1 \\ 0 \\ \dfrac{u}{\sqrt{r^2+u^2+v^2}} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \dfrac{v}{\sqrt{r^2+u^2+v^2}} \end{pmatrix} \\ &= \begin{pmatrix} -\dfrac{u}{\sqrt{r^2+u^2+v^2}} \\ -\dfrac{v}{\sqrt{r^2+u^2+v^2}} \\ 1 \end{pmatrix}\end{aligned}$$

第三成分は 0 ではないから、$\sigma_{+u} \times \sigma_{+v} \neq 0$
よって、σ_{+u} と σ_{+v} は線形独立

$$\begin{pmatrix} u \\ v \\ \sqrt{r^2+u^2+v^2} \end{pmatrix} = \begin{pmatrix} u' \\ v' \\ \sqrt{r^2+u'^2+v'^2} \end{pmatrix}$$

とする. 第一成分と第二成分の比較より、$\begin{cases} u = u' \\ v = v' \end{cases}$

よって、σ_+ は単射である

(3)

$S \subset \sigma_+ \left(\mathbb{R}^2\right) \cup \sigma_- \left(\mathbb{R}^2\right)$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S$ とすると、$x^2+y^2-z^2+r^2=0$

$u^2+v^2-\left(\pm\sqrt{r^2+u^2+v^2}\right)^2 = -r^2$ で、$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \sigma_+ \left(\mathbb{R}^2\right) \cup \sigma_- \left(\mathbb{R}^2\right)$

$S \supset \sigma_+ \left(\mathbb{R}^2\right) \cup \sigma_- \left(\mathbb{R}^2\right)$

$u^2+v^2-\left(\pm\sqrt{r^2+u^2+v^2}\right)^2 = -r^2$ で、$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S$

以上より、$S = \sigma_+ \left(\mathbb{R}^2\right) \cup \sigma_- \left(\mathbb{R}^2\right)$

24　6.1

(1)

$$f = x\sin\frac{z}{k} - y\cos\frac{z}{k}, \nabla f = \begin{pmatrix} \sin\frac{z}{k} \\ -\cos\frac{z}{k} \\ \frac{x}{k}\cos\frac{z}{k} + \frac{y}{k}\sin\frac{z}{k} \end{pmatrix}$$
$$T_pS = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| \begin{pmatrix} \sin\frac{z_0}{k} \\ -\cos\frac{z_0}{k} \\ \frac{x_0}{k}\cos\frac{z_0}{k} + \frac{y_0}{k}\sin\frac{z_0}{k} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$$
$$\iff T_pS = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| x\sin\frac{z_0}{k} - y\cos\frac{z_0}{k} + \left(\frac{x_0}{k}\cos\frac{z_0}{k} + \frac{y_0}{k}\sin\frac{z_0}{k}\right) z = 0 \right\}$$

(2)

$$\mathbf{n}\left(\sigma\left(u,v\right)\right)=\frac{\left(\nabla f\right)\left(\sigma\left(u,v\right)\right)}{\left\|\left(\nabla f\right)\sigma\left(u,v\right)\right\|}=\frac{1}{\sqrt{1+\frac{u^2}{k^2}}}\begin{pmatrix}\sin v\\-\cos v\\\frac{u}{k}\end{pmatrix}=\frac{k}{\sqrt{u^2+k^2}}\begin{pmatrix}\sin v\\-\cos v\\\frac{u}{k}\end{pmatrix}$$

$$\sigma_u=\begin{pmatrix}\cos v\\\sin v\\0\end{pmatrix},\sigma_v=\begin{pmatrix}-u\sin v\\u\cos v\\k\end{pmatrix}$$

$$\begin{aligned}\sigma_u\times\sigma_v&=\begin{pmatrix}\cos v\\\sin v\\0\end{pmatrix}\times\begin{pmatrix}-u\sin v\\u\cos v\\k\end{pmatrix}\\&=\begin{pmatrix}k\sin v\\-k\cos v\\u\end{pmatrix}\end{aligned}$$

$\|\sigma_u\times\sigma_v\|=\sqrt{u^2+k^2}$

だから、$\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}=\frac{1}{\sqrt{u^2+k^2}}\begin{pmatrix}k\sin v\\-k\cos v\\u\end{pmatrix}=\frac{k}{\sqrt{u^2+k^2}}\begin{pmatrix}\sin v\\-\cos v\\\frac{u}{k}\end{pmatrix}$

25 6.2

(1)

$$f=x^2-y^2-r^2,\nabla f=\begin{pmatrix}2x\\-2y\\0\end{pmatrix}$$

$$T_pS=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3\middle|\begin{pmatrix}2x_0\\-2y_0\\0\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=0\right\}=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3\middle|2x_0x-2y_0y=0\right\}$$

(2)

$$\mathbf{n}\left(\sigma_+\left(u,v\right)\right)=\frac{\left(\nabla f\right)\left(\sigma_+\left(u,v\right)\right)}{\left\|\left(\nabla f\right)\left(\sigma_+\left(u,v\right)\right)\right\|}=\frac{1}{2r\sqrt{\cosh 2u}}\begin{pmatrix}2r\cosh u\\-2r\sinh u\\0\end{pmatrix}=\frac{1}{\sqrt{\cosh 2u}}\begin{pmatrix}\cosh u\\-\sinh u\\0\end{pmatrix}$$

$$\sigma_{+_u}=\begin{pmatrix}r\sinh u\\r\cosh u\\0\end{pmatrix},\sigma_{+_v}=\begin{pmatrix}0\\0\\1\end{pmatrix},\sigma_{+_u}\times\sigma_{+_v}=\begin{pmatrix}r\cosh u\\-r\sinh u\\0\end{pmatrix}$$

$$\frac{\sigma_{+_u}\times\sigma_{+_v}}{\|\sigma_{+_u}\times\sigma_{+_v}\|}=\frac{1}{\sqrt{\cosh 2u}}\begin{pmatrix}\cosh u\\-\sinh u\\0\end{pmatrix}=\mathbf{n}\left(\sigma_+\left(u,v\right)\right)$$

よって、σ_+ は正の向きである

$$\mathbf{n}\left(\sigma_-\left(u,v\right)\right)=\frac{\left(\nabla f\right)\left(\sigma_-\left(u,v\right)\right)}{\left\|\left(\nabla f\right)\left(\sigma_-\left(u,v\right)\right)\right\|}=\frac{1}{\sqrt{\cosh 2u}}\begin{pmatrix}-\cosh u\\-\sinh u\\0\end{pmatrix}$$

$$\sigma_{-_u}=\begin{pmatrix}-r\sinh u\\r\cosh u\\0\end{pmatrix},\sigma_{-_v}=\begin{pmatrix}0\\0\\1\end{pmatrix},\sigma_{-_u}\times\sigma_{-_v}=\begin{pmatrix}r\cosh u\\r\sinh u\\0\end{pmatrix}$$

$$\frac{\sigma_{-u}\times\sigma_{-v}}{\|\sigma_{-u}\times\sigma_{-v}\|}=\frac{1}{\sqrt{\cosh 2u}}\begin{pmatrix}\cosh u\\ \sinh u\\ 0\end{pmatrix}=-\mathbf{n}\left(\sigma_{-}\left(u,v\right)\right)$$

だから、σ_{-} は負の向き

26　6.1

(1)

$$f=x^2+y^2-a^2\cosh^2\frac{z}{a},\nabla f=\begin{pmatrix}2x\\ 2y\\ -a\sinh\dfrac{2z}{a}\end{pmatrix}$$

$$T_pS=\left\{\begin{pmatrix}x\\ y\\ z\end{pmatrix}\in\mathbb{R}^3\middle|\begin{pmatrix}2x_0\\ 2y_0\\ -a\sinh\dfrac{2z_0}{a}\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix}=0\right\}$$

$$\Longleftrightarrow\ T_pS=\left\{\begin{pmatrix}x\\ y\\ z\end{pmatrix}\in\mathbb{R}^3\middle|2x_0x+2y_0y-az\sinh\frac{2z_0}{a}=0\right\}$$

(2)

$$\mathbf{n}\left(\sigma\left(u,v\right)\right)=\frac{\left(\nabla f\right)\left(\sigma\left(u,v\right)\right)}{\|\left(\nabla f\right)\left(\sigma\left(u,v\right)\right)\|}=\frac{1}{2a\cosh^2 u}\begin{pmatrix}2\cosh u\cos v\\ 2\cosh u\sin v\\ -a\sinh 2u\end{pmatrix}$$

$$\sigma_u=\begin{pmatrix}a\sinh u\cos v\\ a\sinh u\sin v\\ a\end{pmatrix},\sigma_v=\begin{pmatrix}-a\cosh u\sin v\\ a\cosh u\cos v\\ 0\end{pmatrix}$$

$$\begin{aligned}\sigma_u\times\sigma_v&=\begin{pmatrix}a\sinh u\cos v\\ a\sinh u\sin v\\ a\end{pmatrix}\times\begin{pmatrix}-a\cosh u\sin v\\ a\cosh u\cos v\\ 0\end{pmatrix}\\&=\begin{pmatrix}-a^2\cosh u\cos v\\ -a^2\cosh u\sin v\\ a^2\sinh u\cosh u\end{pmatrix}\end{aligned}$$

$$\|\sigma_u\times\sigma_v\|=a^2\cosh^2 u$$

$$\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}=\frac{1}{a^2\cosh^2 u}\begin{pmatrix}-a^2\cosh u\cos v\\ -a^2\cosh u\sin v\\ a^2\sinh u\cosh u\end{pmatrix}=-\mathbf{n}\left(\sigma\left(u,v\right)\right)$$

よって、σ は負の向き

27　6.2

(1)

$$f = x^2 + y^2 - z^2 - r^2, \nabla f = \begin{pmatrix} 2x \\ 2y \\ -2z \end{pmatrix}$$

$$T_pS = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| \begin{pmatrix} 2x_0 \\ 2y_0 \\ -2z_0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| x_0 x + y_0 y - z_0 z = 0 \right\}$$

(2)

$$\mathbf{n}\left(\sigma\left(u,v\right)\right) = \frac{\left(\nabla f\right)\left(\sigma\left(u,v\right)\right)}{\left\|\left(\nabla f\right)\left(\sigma\left(u,v\right)\right)\right\|} = \frac{1}{\sqrt{\cosh 2u}} \begin{pmatrix} \cosh u \cos v \\ \cosh u \sin v \\ -\sinh u \end{pmatrix}$$

$$\sigma_u = \begin{pmatrix} r \sinh u \cos v \\ r \sinh u \sin v \\ r \cosh u \end{pmatrix}, \sigma_v = \begin{pmatrix} -r \cosh u \sin v \\ r \cosh u \cos v \\ 0 \end{pmatrix}, \sigma_u \times \sigma_v = \begin{pmatrix} -r^2 \cosh^2 u \cos v \\ -r^2 \cosh^2 u \sin v \\ r^2 \sinh u \cosh u \end{pmatrix}$$

$$\frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{r^2 \cosh u \sqrt{\cosh 2u}} \begin{pmatrix} -r^2 \cosh^2 u \cos v \\ -r^2 \cosh^2 u \sin v \\ r^2 \sinh u \cosh u \end{pmatrix} = \frac{1}{\sqrt{\cosh 2u}} \begin{pmatrix} -\cosh u \cos v \\ -\cosh u \sin v \\ \sinh u \end{pmatrix}$$
$$= -\mathbf{n}\left(\sigma\left(u,v\right)\right)$$

28　6.3

(1)

$$f = x^2 + y^2 - z^2 + r^2, \nabla f = \begin{pmatrix} 2x \\ 2y \\ -2z \end{pmatrix}$$

$$T_pS = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| \begin{pmatrix} 2x_0 \\ 2y_0 \\ -2z_0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| x_0 x + y_0 y - z_0 z = 0 \right\}$$

(2)

$$\mathbf{n}\left(\sigma_+\left(u,v\right)\right) = \frac{1}{\sqrt{r^2 + 2u^2 + 2v^2}} \begin{pmatrix} u \\ v \\ -\sqrt{r^2 + u^2 + v^2} \end{pmatrix}$$

$$\mathbf{n}\left(\sigma_-\left(u,v\right)\right) = \frac{1}{\sqrt{r^2 + 2u^2 + 2v^2}} \begin{pmatrix} u \\ v \\ \sqrt{r^2 + u^2 + v^2} \end{pmatrix}$$

$$\sigma_{+_u} = \begin{pmatrix} 1 \\ 0 \\ \frac{u}{\sqrt{r^2 + u^2 + v^2}} \end{pmatrix}, \sigma_{+_v} = \begin{pmatrix} 0 \\ 1 \\ \frac{v}{\sqrt{r^2 + u^2 + v^2}} \end{pmatrix}, \sigma_{+_u} \times \sigma_{+_v} = \begin{pmatrix} -\frac{u}{\sqrt{r^2+u^2+v^2}} \\ -\frac{v}{\sqrt{r^2+u^2+v^2}} \\ 1 \end{pmatrix}$$

$$\sigma_{-_u} = \begin{pmatrix} 1 \\ 0 \\ -\frac{u}{\sqrt{r^2 + u^2 + v^2}} \end{pmatrix}, \sigma_{-_v} = \begin{pmatrix} 0 \\ 1 \\ -\frac{v}{\sqrt{r^2 + u^2 + v^2}} \end{pmatrix}, \sigma_{-_u} \times \sigma_{-_v} = \begin{pmatrix} \frac{u}{\sqrt{r^2+u^2+v^2}} \\ \frac{v}{\sqrt{r^2+u^2+v^2}} \\ 1 \end{pmatrix}$$

$$\frac{\sigma_{+u}\times\sigma_{+v}}{\|\sigma_{+u}\times\sigma_{+v}\|}=\frac{1}{\sqrt{r^2+2u^2+2v^2}}\begin{pmatrix}-u\\-v\\\sqrt{r^2+u^2+v^2}\end{pmatrix}=-\mathbf{n}\left(\sigma_+\left(u,v\right)\right)$$

$$\frac{\sigma_{-u}\times\sigma_{-v}}{\|\sigma_{-u}\times\sigma_{-v}\|}=\frac{1}{\sqrt{r^2+2u^2+2v^2}}\begin{pmatrix}u\\v\\\sqrt{r^2+u^2+v^2}\end{pmatrix}=\mathbf{n}\left(\sigma_-\left(u,v\right)\right)$$

よって、正になるものは σ_-

29　7.1

(i)

$f\left(x,y,z\right)=\sqrt{x^2+y^2}, T_1=\sigma\left(\left(0,k\right)\times\left(0,2\pi\right)\right)$

$(M1)$　$T=T_1$

$(M2)$　σ 局所パラメーター表示、$(0,k)\times(0,2\pi)$：面積確定、有界
$T_1=\sigma\left(\left(0,2k\right)\times\left(0,2\pi\right)\right)$ より OK

$(M3)$　$i\neq j, i,j\in\{1\}$ は取れない. よって、OK

また $\sigma_u\times\sigma_v=\begin{pmatrix}k\sin v\\k\cos v\\u\end{pmatrix}, \|\sigma_u\times\sigma_v\|=\sqrt{u^2+k^2}$

よって

$$\begin{aligned}\iint_T \mathrm{d}A&=\iint_{(0,k)\times(0,2\pi)} f\left(\sigma\left(u,v\right)\right)\|\sigma_u\times\sigma_v\|\,\mathrm{d}u\mathrm{d}v\\&=\int_0^{2\pi}\int_0^k u\sqrt{u^2+k^2}\mathrm{d}u\mathrm{d}v\\&=\frac{2}{3}\pi\left(2\sqrt{2}k^3-k^3\right)\\&=\frac{2}{3}\pi k^3\left(2\sqrt{2}-1\right)\end{aligned}$$

(ii)

$$\begin{aligned}Area\left(T\right)&=\iint_{(0,k)\times(0,2\pi)}\sqrt{u^2+k^2}\mathrm{d}u\mathrm{d}v\\&=\int_0^{2\pi}\int_0^k\sqrt{u^2+k^2}\mathrm{d}u\mathrm{d}v\\&=2\pi k^2\int_0^{\frac{\pi}{4}}\frac{1}{\cos^3\theta}\mathrm{d}\theta\\&=\left(\sqrt{2}+\log\left(\sqrt{2}+1\right)\right)\pi k^2\end{aligned}$$

(iii)

$$\begin{aligned}\iint_T v\cdot \mathrm{d}A &= \iint_{(0,k)\times(0,2\pi)} v\left(\sigma\left(u,v\right)\right)\cdot\left(\sigma_u\times\sigma_v\right)\mathrm{d}u\mathrm{d}v\\ &= \int_0^{2\pi}\int_0^k \left(-ku\cos 2v + k^2uv^2\right)\mathrm{d}u\mathrm{d}v\\ &= \frac{4}{3}\pi^3k^4\end{aligned}$$

30　7.2

(i)

$(M1)$ $S^2\left(1\right)=\sigma\left(\left(0,\pi\right)\times\left(0,2\pi\right)\right)\cup\tau\left(\left\{\frac{\pi}{2}\right\}\times\left[\frac{\pi}{2},\frac{3}{2}\pi\right]\right)$,OK

$(M2)$ $\left(0,\pi\right)\times\left(0,2\pi\right)$ と $\left\{\frac{\pi}{2}\right\}\times\left[\frac{\pi}{2},\frac{3}{2}\pi\right]$ は面積確定な有界集合

$(M3)$ $\sigma^{-1}\left(\sigma\left(\left(0,\pi\right)\times\left(0,2\pi\right)\right)\cap\tau\left(\left\{\frac{\pi}{2}\right\}\times\left[\frac{\pi}{2},\frac{3}{2}\pi\right]\right)\right)=\sigma^{-1}\left(\emptyset\right)$ は面積 0

$$\sigma_u\times\sigma_v=\begin{pmatrix}\sin^2 u\cos v\\ \sin^2 u\sin v\\ \cos u\sin u\end{pmatrix}=\sin u\sigma\left(u,v\right),\left\|\sigma\left(u,v\right)\right\|=\sin u$$

$\dfrac{\sigma_u\times\sigma_v}{\left\|\sigma_u\times\sigma_v\right\|}=\sigma\left(u,v\right)=n\left(\sigma\left(u,v\right)\right)$

同様に $\dfrac{\tau_u\times\tau_v}{\left\|\tau_u\times\tau_v\right\|}=\tau\left(u,v\right)=n\left(\tau\left(u,v\right)\right)$

よって、σ,τ も正の向き

$$\begin{aligned}\iint_{S^2(1)} v\cdot\mathrm{d}A &= \iint_{(0,\pi)\times(0,2\pi)} v\left(\sigma\left(u,v\right)\right)\cdot\left(\sigma_u\times\sigma_v\right)\mathrm{d}u\mathrm{d}v+\iint_{\left\{\frac{\pi}{2}\right\}\times\left[\frac{\pi}{2},\frac{3}{2}\pi\right]} v\left(\tau\left(u,v\right)\right)\cdot\left(\tau_u\times\tau_v\right)\mathrm{d}u\mathrm{d}v\\ &= 4\pi\end{aligned}$$

(ii)

$\nabla\cdot\omega=v$ なら、$\displaystyle\iint_{S^2(1)}\left(\nabla\cdot\omega\right)\mathrm{d}A=0$

(i) の結論と反する、このような ω は存在しない

31　7.1

(i)

$\sigma(u,v) = \begin{pmatrix} a\cosh u \cos v \\ a\cosh u \sin v \\ au \end{pmatrix}$ から、$T = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S \middle| -1 < z < 1 \right\}$ より、$u \in \left(-\frac{1}{a}, \frac{1}{a}\right)$

$$\begin{aligned} Area(T) &= \iint_{\left(-\frac{1}{a},\frac{1}{a}\right)\times[0,2\pi]} \|\sigma_u \times \sigma_v\| \,\mathrm{d}u\mathrm{d}v \\ &= \int_{-\frac{1}{a}}^{\frac{1}{a}} \int_0^{2\pi} a^2 \cosh^2 u \mathrm{d}v\mathrm{d}u \\ &= 2a^2\pi \int_{-\frac{1}{a}}^{\frac{1}{a}} \cosh^2 u \mathrm{d}u \\ &= 2a^2\pi \left(\frac{1}{a} + \frac{1}{2}\sinh\frac{2}{a}\right) \\ &= 2a\pi + a^2\pi \sinh\frac{2}{a} \end{aligned}$$

(ii)

$$\begin{aligned} \iint_T v\cdot \mathrm{d}A &= \iint_{\left(-\frac{1}{a},\frac{1}{a}\right)\times[0,2\pi]} \begin{pmatrix} a\cosh u \sin v \\ -a\cosh u\cos v \\ a^2u^2 \end{pmatrix} \cdot \begin{pmatrix} -a^2\cosh u \cos v \\ -a^2\cosh u \sin v \\ a^2 \sinh u \cosh u \end{pmatrix} \mathrm{d}u\mathrm{d}v \\ &= \iint_{\left(-\frac{1}{a},\frac{1}{a}\right)\times[0,2\pi]} a^4u^2 \sinh u \cosh u \mathrm{d}u\mathrm{d}v \\ &= a^4 \int_{-\frac{1}{a}}^{\frac{1}{a}} \int_0^{2\pi} u^2 \sinh u \cosh u \mathrm{d}v\mathrm{d}u \\ &= 2a^4\pi \int_{-\frac{1}{a}}^{\frac{1}{a}} u^2 \sinh u \cosh u \mathrm{d}u \\ &= \frac{1}{4}a^4\pi \left[\left(2u^2+1\right)\cosh 2u - 2u\sinh 2u\right]_{-\frac{1}{a}}^{\frac{1}{a}} \\ &= 0 \end{aligned}$$

32　7.2

(i)

$T_{R,r} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| \left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = r^2 \right\}$

$z^2 = r^2 - \left(\sqrt{x^2+y^2} - R\right)^2 \geq 0$ から $R - r \leq \sqrt{x^2+y^2} \leq R + r$

よって、$\|\mathbf{x}\| = \sqrt{x^2+y^2+z^2} \leq \sqrt{(R+r)^2 + z^2} \leq \sqrt{(R+r)^2 + r^2}$

言い換えれば、$T_{R,r}$ は有界集合である.

また、$f(x,y,z) = \left(\sqrt{x^2+y^2} - R\right)^2 + z^2 - r^2$ とすると、$T_{R,r}$ は $f(x,y,z) = 0$ の開集合 $f^{-1}(\{0\})$ であるから、閉集合である

(ii)

$$\sigma = \begin{pmatrix} (R + r\cos u)\cos v \\ (R + r\cos u)\sin v \\ r\sin u \end{pmatrix}, \sigma_u = \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix}, \sigma_v = \begin{pmatrix} -(R + r\cos u)\sin v \\ (R + r\cos u)\cos v \\ 0 \end{pmatrix}$$

$$\begin{aligned} \sigma_u \times \sigma_v &= \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix} \times \begin{pmatrix} -(R + r\cos u)\sin v \\ (R + r\cos u)\cos v \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -r(R + r\cos u)\cos u\cos v \\ -r(R + r\cos u)\cos u\sin v \\ -r(R + r\cos u)\sin u \end{pmatrix} \end{aligned}$$

$\|\sigma_u \times \sigma_v\| = r(R + r\cos u)$

$$\begin{aligned} Area(T) &= \iint_{[0,2\pi]\times[0,2\pi]} \|\sigma_u \times \sigma_v\| \, \mathrm{d}u\mathrm{d}v \\ &= \int_0^{2\pi}\int_0^{2\pi} r(R + r\cos u)\,\mathrm{d}v\mathrm{d}u \\ &= 2\pi\int_0^{2\pi} r(R + r\cos u)\,\mathrm{d}u \\ &= 4\pi^2 rR \end{aligned}$$

(iii)

$$\sigma(u, v) = \begin{pmatrix} (R + r\cos u)\cos v \\ (R + r\cos u)\sin v \\ r\sin u \end{pmatrix}$$

$$\sigma_u = \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix}, \sigma_v = \begin{pmatrix} -(R + r\cos u)\sin v \\ (R + r\cos u)\cos v \\ 0 \end{pmatrix}$$

$$\begin{aligned} \sigma_u \times \sigma_v &= \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix} \times \begin{pmatrix} -(R + r\cos u)\sin v \\ (R + r\cos u)\cos v \\ 0 \end{pmatrix} && (1) \\ &= \begin{pmatrix} -r(R + r\cos u)\cos u\cos v \\ -r(R + r\cos u)\cos u\sin v \\ -r(R + r\cos u)\sin u\cos^2 v - r(R + r\cos u)\sin u\sin^2 v \end{pmatrix} && (2) \\ &= \begin{pmatrix} -r(R + r\cos u)\cos u\cos v \\ -r(R + r\cos u)\cos u\sin v \\ -r(R + r\cos u)\sin u \end{pmatrix} && (3) \end{aligned}$$

また

$$\begin{aligned}
v\left(\sigma\left(u,v\right)\right) &= v\left(\left(R+r\cos u\right)\cos v, \left(R+r\cos u\right)\sin v, r\sin u\right) && (4)\\
&= \begin{pmatrix} -\dfrac{\left(R+r\cos u\right)\sin v}{\left(R+r\cos u\right)^2} \\ \dfrac{\left(R+r\cos u\right)\cos v}{\left(R+r\cos u\right)^2} \\ \dfrac{r\sin u}{\left(R+r\cos u\right)^2} \end{pmatrix} && (5)\\
&= \begin{pmatrix} -\dfrac{\sin v}{R+r\cos u} \\ \dfrac{\cos v}{R+r\cos u} \\ \dfrac{r\sin u}{\left(R+r\cos u\right)^2} \end{pmatrix} && (6)
\end{aligned}$$

以上

$$\begin{aligned}
\iint_S v\cdot \mathrm{d}A &= \iint_{[0,2\pi]^2} \begin{pmatrix} -\dfrac{\sin v}{R+r\cos u} \\ \dfrac{\cos v}{R+r\cos u} \\ \dfrac{r\sin u}{\left(R+r\cos u\right)^2} \end{pmatrix} \cdot \begin{pmatrix} -r\left(R+r\cos u\right)\cos u\cos v \\ -r\left(R+r\cos u\right)\cos u\sin v \\ -r\left(R+r\cos u\right)\sin u \end{pmatrix} \mathrm{d}u\mathrm{d}v\\
&= \int_0^{2\pi}\int_0^{2\pi}\left(r\cos u\sin v\cos v - r\cos u\sin v\cos v - \frac{r^2\sin^2 u}{R+r\cos u}\right)\mathrm{d}u\mathrm{d}v\\
&= -r^2\int_0^{2\pi}\int_0^{2\pi}\frac{\sin^2 u}{R+r\cos u}\mathrm{d}u\mathrm{d}v\\
&= -2\pi r^2\int_0^{2\pi}\frac{\sin^2 u}{R+r\cos u}\mathrm{d}u
\end{aligned}$$

ここで、$\displaystyle\int_0^{2\pi}\frac{\sin^2 u}{R+r\cos u}\mathrm{d}u$ だけ考えよう

$$\begin{aligned}
\int_0^{2\pi}\frac{\sin^2 u}{R+r\cos u}\mathrm{d}u &= \int_0^{2\pi}\frac{1-\cos^2 u}{R+r\cos u}\mathrm{d}u && (7)\\
&= \int_0^{2\pi}\frac{\frac{1}{r^2}\left(R+r\cos u\right)\left(R-r\cos u\right)+1-\frac{R^2}{r^2}}{R+r\cos u}\mathrm{d}u && (8)\\
&= \int_0^{2\pi}\left(\frac{1}{r^2}\left(R-r\cos u\right)+\frac{1-\frac{R^2}{r^2}}{R+r\cos u}\right)\mathrm{d}u && (9)\\
&= \int_0^{2\pi}\left(\frac{R}{r^2}-\frac{1}{r}\cos u\right)\mathrm{d}u + \frac{r^2-R^2}{r^2}\int_0^{2\pi}\frac{1}{R+r\cos u}\mathrm{d}u && (10)\\
&= \frac{2\pi R}{r^2}+\frac{r^2-R^2}{r^2}\int_0^{2\pi}\frac{1}{R+r\cos u}\mathrm{d}u && (11)
\end{aligned}$$

ここで、$\displaystyle\int_0^{2\pi}\frac{1}{R+r\cos u}\mathrm{d}u$ を計算しよう

$$\int_0^{2\pi}\frac{1}{R+r\cos u}\mathrm{d}u=\int_0^{2\pi}\frac{R-r\cos u}{R^2-r^2\cos^2 u}\mathrm{d}u \tag{12}$$

$$=R\int_0^{2\pi}\frac{1}{R^2-r^2\cos^2 u}\mathrm{d}u-r\int_0^{2\pi}\frac{\cos u}{R^2-r^2\cos^2 u}\mathrm{d}u \tag{13}$$

$$=R\int_0^{2\pi}\frac{1}{R^2-r^2\cos^2 u}\mathrm{d}u \tag{14}$$

後ろの $\displaystyle\int_0^{2\pi}\frac{\cos u}{R^2-r^2\cos^2 u}\mathrm{d}u$ について、$\displaystyle\frac{\cos u}{R^2-r^2\cos^2 u}$ は奇関数だから積分範囲内で総和 0

$$\int_0^{2\pi}\frac{1}{R^2-r^2\cos^2 u}\mathrm{d}u=\int_0^{2\pi}\frac{1}{R^2-r^2\frac{\cos 2u+1}{2}}\mathrm{d}u \tag{15}$$

$$=\int_0^{2\pi}\frac{1}{\left(R^2-\frac{r^2}{2}\right)-\frac{r^2}{2}\cos 2u}\mathrm{d}u \tag{16}$$

計算便利のため、$a=R^2-\dfrac{r^2}{2}, b=\dfrac{r^2}{2}$ とすると (28) 式は $\displaystyle\int_0^{2\pi}\frac{1}{a-b\cos 2u}\mathrm{d}u$ になり.
また、定積分の処理が面倒なので、不定積分の形で計算しよう

$$\int\frac{1}{a-b\cos 2u}\mathrm{d}u=\frac{1}{2}\int\frac{1}{a-b\cos x}\mathrm{d}x \tag{17}$$

ここで、$t=\tan\dfrac{x}{2}$ と変換すると $\mathrm{d}x=\dfrac{2}{1+t^2}\mathrm{d}t, \cos x=\dfrac{1-t^2}{1+t^2}$ で
積分の上下限は共に 0 になるが、実際 2 回 $-\infty\to\infty$ の広義積分が出てくるから

$$\frac{1}{2}\int\frac{1}{a-b\cos x}\mathrm{d}x=2\int\frac{1}{2}\cdot\frac{1}{a-b\cdot\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\mathrm{d}t \tag{18}$$

$$=2\int\frac{1}{a\left(1+t^2\right)-b\left(1-t^2\right)}\mathrm{d}t \tag{19}$$

$$=2\int\frac{1}{(a-b)+(a+b)\,t^2}\mathrm{d}t \tag{20}$$

$$=\frac{2}{a-b}\int\frac{1}{1+\frac{a+b}{a-b}t^2}\mathrm{d}t \tag{21}$$

そして、$\dfrac{a+b}{a-b}t^2 = \tan^2\theta$ となる変数変換をすると $\mathrm{d}t = \dfrac{1}{\cos^2\theta}\sqrt{\dfrac{a-b}{a+b}}\mathrm{d}\theta$ となり

$$\begin{aligned}
\frac{2}{a-b}\int \frac{1}{1+\frac{a+b}{a-b}t^2}\mathrm{d}t &= \frac{2}{a-b}\int \frac{1}{1+\tan^2\theta}\frac{1}{\cos^2\theta}\sqrt{\frac{a-b}{a+b}}\mathrm{d}\theta && (22)\\
&= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\int \mathrm{d}\theta && (23)\\
&= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\theta && (24)\\
&= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}t\right) && (25)\\
&= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}\tan\frac{x}{2}\right) && (26)\\
&= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}\tan u\right) && (27)
\end{aligned}$$

以上より

$$\int \frac{1}{a-b\cos 2u}\mathrm{d}u = \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}\tan u\right) + C \qquad (28)$$

から

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{a-b\cos 2u}\mathrm{d}u &= \lim_{\epsilon\to 0+}\left[\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}\tan u\right)\right]_0^{\frac{\pi}{2}-\epsilon} && (29)\\
&\quad + \lim_{\epsilon\to 0+}\left[\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}\tan u\right)\right]_{\frac{\pi}{2}+\epsilon}^{\frac{3\pi}{2}-\epsilon} && (30)\\
&\quad + \lim_{\epsilon\to 0+}\left[\frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\arctan\left(\sqrt{\frac{a+b}{a-b}}\tan u\right)\right]_{\frac{3\pi}{2}+\epsilon}^{2\pi} && (31)\\
&= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\cdot\frac{\pi}{2}\cdot 2 && (32)\\
&= \frac{2\pi}{a-b}\sqrt{\frac{a-b}{a+b}} && (33)
\end{aligned}$$

よって

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{R+r\cos u}\mathrm{d}u &= R\int_0^{2\pi}\frac{1}{a-b\cos 2u}\mathrm{d}u && (34)\\
&= \frac{2\pi R}{a-b}\sqrt{\frac{a-b}{a+b}} && (35)\\
&= \frac{2\pi R}{R^2-r^2}\sqrt{\frac{R^2-r^2}{R^2}} && (36)
\end{aligned}$$

(24) 式に代入すると

$$\int_0^{2\pi}\frac{\sin^2 u}{R+r\cos u}\mathrm{d}u = \frac{2\pi R}{r^2}-\frac{R^2-r^2}{r^2}\int_0^{2\pi}\frac{1}{R+r\cos u}\mathrm{d}u \tag{37}$$
$$= \frac{2\pi R}{r^2}-\frac{R^2-r^2}{r^2}\frac{2\pi R}{R^2-r^2}\sqrt{\frac{R^2-r^2}{R^2}} \tag{38}$$
$$= \frac{2\pi R}{r^2}-\frac{2\pi R}{r^2}\sqrt{\frac{R^2-r^2}{R^2}} \tag{39}$$
$$= \frac{2\pi R}{r^2}-\frac{2\pi}{r^2}\sqrt{R^2-r^2} \tag{40}$$
$$= \frac{2\pi}{r^2}\left(R-\sqrt{R^2-r^2}\right) \tag{41}$$

だから

$$\iint_S v\cdot\mathrm{d}A = -2\pi r^2\int_0^{2\pi}\frac{\sin^2 u}{R+r\cos u}\mathrm{d}u \tag{42}$$
$$= -2\pi r^2\cdot\frac{2\pi}{r^2}\left(R-\sqrt{R^2-r^2}\right) \tag{43}$$
$$= -4\pi^2\left(R-\sqrt{R^2-r^2}\right) \tag{44}$$

(iv)

$v=\nabla\times\omega$ をみたす ω が存在すると仮定すると、Stokes の定理より

$$\iint_S v\cdot\mathrm{d}A = \iint_S (\nabla\times\omega)\cdot\mathrm{d}A = 0 \tag{45}$$

(2) の計算結果により、$R=\sqrt{R^2-r^2}$. 言い換えれば、$r=0$
これは $r>0$ と反するから. このような ω は存在しない

33　8.1

(1)

$$\partial V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R} \middle| \left(x^2+y^2+z^2=r^2\right) \vee \left(x^2+y^2+z^2=4r^2\right) \right\} = S^2(r)\cup S^2(2r)$$

(2)

単位法ベクトル場 ω は $\mathbf{p}\in S^2(a)$ に対し、$\omega(\mathbf{p})=\dfrac{1}{a}\mathbf{p}$ となるので

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \partial V \text{ に対して } \mathbf{r}(x,y,z) = \begin{cases} \dfrac{1}{2r}\begin{pmatrix} x \\ y \\ z \end{pmatrix} & x^2+y^2+z^2=4r^2 \\ -\dfrac{1}{r}\begin{pmatrix} x \\ y \\ z \end{pmatrix} & x^2+y^2+z^2=r^2 \end{cases}$$

(3)

$\nabla \cdot \mathbf{v} = 1 + 1 + 0 = 2$

(4)

$$\begin{aligned}
\iiint_{\overline{V}} (\nabla \cdot \mathbf{v})\, \mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_r^{2r} \int_0^{\pi} \int_0^{2\pi} 2\rho^2 \sin\phi \mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\rho \\
&= 4\pi \int_r^{2r} \int_0^{\pi} \rho^2 \sin\phi \mathrm{d}\phi\mathrm{d}\rho \\
&= 4\pi \int_r^{2r} \rho^2 \left[-\cos\phi\right]_0^{\pi} \mathrm{d}\rho \\
&= 8\pi \int_r^{2r} \rho^2 \mathrm{d}\rho \\
&= 8\pi \left[\frac{1}{3}\rho^3\right]_r^{2r} \\
&= \frac{56}{3}\pi r^3
\end{aligned}$$

$$\sigma(\rho', u, v) = \rho' \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix}, \sigma_u = \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ -\sin u \end{pmatrix}, \sigma_v = \begin{pmatrix} -\sin u \sin v \\ \sin u \cos v \\ 0 \end{pmatrix}$$

$$\begin{aligned}
\sigma_u \times \sigma_v &= \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ -\sin u \end{pmatrix} \times \begin{pmatrix} -\sin u \sin v \\ \sin u \cos v \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \sin^2 u \cos v \\ \sin^2 u \sin v \\ \sin u \cos u \end{pmatrix} \\
&= \rho' \sin u \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\iint_{\partial V} \mathbf{v} \cdot \mathrm{d}\mathbf{A} &= \int_0^{\pi} \int_0^{2\pi} 2r \cdot 2r \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ 0 \end{pmatrix} \cdot 2r \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix} \sin u \mathrm{d}v\mathrm{d}u \\
&- \int_0^{\pi} \int_0^{2\pi} r \cdot r \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ 0 \end{pmatrix} \cdot r \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix} \sin u \mathrm{d}v\mathrm{d}u \\
&= \int_0^{\pi} \int_0^{2\pi} 7r^3 \sin^3 u \mathrm{d}v\mathrm{d}u \\
&= 14\pi r^3 \int_0^{\pi} \sin^3 u \mathrm{d}u \\
&= \frac{56}{3}\pi r^3
\end{aligned}$$

34　8.2

$T=\left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\in\mathbb{R}^3\middle| x\geq 0, y\geq 0, z\geq 0, x+y+z\leq 2\right\}, \nabla\cdot\mathbf{v}=2\left(x+y+z\right)$

$$\begin{aligned}
\iint_{\partial T}\mathbf{v}\cdot\mathrm{d}\mathbf{A} &= \int_0^2\int_0^{2-x}\int_0^{2-x-y} 2\left(x+y+z\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= 2\int_0^2\int_0^{2-x}\left[\left(x+y\right)z+\frac{1}{2}z^2\right]_0^{2-x-y}\mathrm{d}y\mathrm{d}x \\
&= 2\int_0^2\int_0^{2-x}\left(\left(x+y\right)\left(2-x-y\right)+\frac{1}{2}\left(2-x-y\right)^2\right)\mathrm{d}y\mathrm{d}x \\
&= \int_0^2\int_0^{2-x}\left(4-x^2-2xy-y^2\right)\mathrm{d}y\mathrm{d}x \\
&= \int_0^2\left[\left(4-x^2\right)y-xy^2-\frac{1}{3}y^3\right]_0^{2-x}\mathrm{d}x \\
&= \int_0^2\left(\left(2+x\right)\left(2-x\right)^2-x\left(2-x\right)^2-\frac{1}{3}\left(2-x\right)^3\right)\mathrm{d}x \\
&= \frac{1}{3}\int_0^2\left(x-2\right)^2\left(x+4\right)\mathrm{d}x \\
&= \frac{1}{3}\left[\frac{x^4}{4}-6x^2+16x\right]_0^2 \\
&= \frac{1}{3}\cdot 12 = 4
\end{aligned}$$

35　8.1

(1)

$\partial V=\left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\in\mathbb{R}^3\middle|\frac{x^2}{4}+y^2+z^2=1\right\}$

(2)

$f=\frac{x^2}{4}+y^2+z^2-1, \nabla f=\begin{pmatrix} \frac{x}{2} \\ 2y \\ 2z \end{pmatrix}$

$\mathbf{n}=\frac{1}{\sqrt{\frac{x^2}{4}+y^2+z^2}}\begin{pmatrix} \frac{x}{2} \\ 2y \\ 2z \end{pmatrix}$

(3)

$\nabla\cdot\mathbf{v}=1+1+1=3$

(4)

$$\begin{aligned}\iint_{\partial V}(\nabla\cdot\mathbf{v})\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1\int_0^{2\pi}\int_0^{\pi}3\rho^2\sin\phi\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}\rho \\ &= 6\int_0^1\int_0^{2\pi}\rho^2\mathrm{d}\theta\mathrm{d}\rho \\ &= 12\pi\int_0^1\rho^2\mathrm{d}\rho \\ &= 4\pi\end{aligned}$$

$$\sigma = \begin{pmatrix}2\sin\phi\cos\theta\\ \sin\phi\sin\theta\\ \cos\phi\end{pmatrix}, \sigma_\phi = \begin{pmatrix}2\cos\phi\cos\theta\\ \cos\phi\sin\theta\\ -\sin\phi\end{pmatrix}, \sigma_\theta = \begin{pmatrix}-2\sin\phi\sin\theta\\ \sin\phi\cos\theta\\ 0\end{pmatrix}$$

$$\begin{aligned}\sigma_\phi\times\sigma_\theta &= \begin{pmatrix}2\cos\phi\cos\theta\\ \cos\phi\sin\theta\\ -\sin\phi\end{pmatrix}\times\begin{pmatrix}-2\sin\phi\sin\theta\\ \sin\phi\cos\theta\\ 0\end{pmatrix} \\ &= \begin{pmatrix}\sin^2\phi\cos\theta\\ 2\sin^2\phi\sin\theta\\ 2\sin\phi\cos\phi\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\iint_{\partial V}\mathbf{v}\cdot\mathrm{d}\mathbf{A} &= \int_0^{2\pi}\int_0^{\pi}\begin{pmatrix}2\sin\phi\cos\theta\\ \sin\phi\sin\theta\\ \cos\phi\end{pmatrix}\cdot\begin{pmatrix}\sin^2\phi\cos\theta\\ 2\sin^2\phi\sin\theta\\ 2\sin\phi\cos\phi\end{pmatrix}\mathrm{d}\phi\mathrm{d}\theta \\ &= 2\int_0^{2\pi}\int_0^{\pi}\left(\sin^3\phi+\sin\phi\cos^2\phi\right)\mathrm{d}\phi\mathrm{d}\theta \\ &= 2\pi\int_0^{\pi}\sin\phi\mathrm{d}\theta \\ &= 4\pi\end{aligned}$$

36　8.2

$\nabla\cdot\mathbf{v} = y+0+2y = 3y$

$$\begin{aligned}\iint_{\partial T}\mathbf{v}\cdot\mathrm{d}\mathbf{A} &= \iiint_{\overline{T}}3y\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int_0^1\int_0^{1-x}\int_0^{1-x-y}3y\mathrm{d}z\mathrm{d}y\mathrm{d}x \\ &= \int_0^1\int_0^{1-x}3y\left(1-x-y\right)\mathrm{d}y\mathrm{d}x \\ &= \frac{1}{2}\int_0^1(1-x)^3\,\mathrm{d}x \\ &= \frac{1}{8}\end{aligned}$$

37 8.3

$$\sigma = r \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix}, \sigma_u = \begin{pmatrix} r\cos u \cos v \\ r\cos u \sin v \\ -r\sin u \end{pmatrix}, \sigma_v = \begin{pmatrix} -r\sin u \sin v \\ r\sin u \cos v \\ 0 \end{pmatrix}$$

$$\begin{aligned} \sigma_u \times \sigma_v &= \begin{pmatrix} r\cos u \cos v \\ r\cos u \sin v \\ -r\sin u \end{pmatrix} \times \begin{pmatrix} -r\sin u \sin v \\ r\sin u \cos v \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} r^2 \sin^2 u \cos v \\ r^2 \sin^2 u \sin v \\ r^2 \sin u \cos u \end{pmatrix} \end{aligned}$$

(1)

$$\begin{aligned} \iint_{S^2(r)} \mathbf{v} \cdot \mathrm{d}\mathbf{A} &= \iiint_{\overline{S^2(r)}} 3\left(x^2+y^2+z^2\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= 3\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} \left(x^2+y^2+z^2\right) \mathrm{d}z\mathrm{d}y\mathrm{d}x \\ &= 3\int_0^r \int_0^{\sqrt{r^2-x^2}} \left(\left(x^2+y^2\right)\sqrt{r^2-x^2-y^2} + \frac{1}{3}\left(r^2-x^2-y^2\right)^{\frac{3}{2}}\right) \mathrm{d}y\mathrm{d}x \\ &= \frac{3}{8}\pi \int_0^r \left(r^4-x^4\right) \mathrm{d}x \\ &= \frac{3}{8}\pi \left[r^4 x - \frac{1}{5}x^5\right]_0^r \\ &= \frac{3}{10}\pi r^5 \end{aligned}$$

(2)

$$\begin{aligned} \iint_{S^2(r)} \mathbf{v} \cdot \mathrm{d}\mathbf{A} &= \iiint_{\overline{S^2(r)}} \left(x^2+y^2+z^2\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \frac{1}{3}\iiint_{\overline{S^2(r)}} 3\left(x^2+y^2+z^2\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \frac{1}{10}\pi r^5 \end{aligned}$$

(3)

$\nabla \cdot \mathbf{v} = 3 + x + 2y$

$$\begin{aligned}
\iint_{S^2(r)} \mathbf{v} \cdot \mathrm{d}\mathbf{A} &= \iiint_{\overline{S^2(r)}} (3 + x + 2y)\, \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} (3 + x + 2y)\, \mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \int_0^r \int_0^{\sqrt{r^2-x^2}} (3 + x + 2y) \sqrt{r^2 - x^2 - y^2} \mathrm{d}y\mathrm{d}x \\
&= \int_0^r \left(\frac{1}{4}\pi (x+3) \left(r^2 - x^2\right) + \frac{2}{3} \left(r^2 - x^2\right)^{\frac{3}{2}} \right) \mathrm{d}x \\
&= \frac{1}{16}\pi r^3 (3r + 8)
\end{aligned}$$

38　8.4

$$\sigma = \begin{pmatrix} (R + r\cos u)\cos v \\ (R + r\cos u)\sin v \\ r\sin u \end{pmatrix}, \sigma_u = \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix}, \sigma_v = \begin{pmatrix} -(R + r\cos u)\sin v \\ (R + r\cos u)\cos v \\ 0 \end{pmatrix}$$

$$\begin{aligned}
\sigma_u \times \sigma_v &= \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix} \times \begin{pmatrix} -(R + r\cos u)\sin v \\ (R + r\cos u)\cos v \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -r(R + r\cos u)\cos u\cos v \\ -r(R + r\cos u)\cos u\sin v \\ -r(R + r\cos u)\sin u \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\iint_{T_{R,r}} \mathbf{v} \cdot \mathrm{d}\mathbf{A} &= \iiint_{\overline{T_{R,r}}} 2\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= 2\int_0^r \int_0^{2\pi} \int_0^{2\pi} \rho (R + \rho\cos u)\, \mathrm{d}v\mathrm{d}u\mathrm{d}\rho \\
&= 4\pi \int_0^r \int_0^{2\pi} \rho (R + \rho\cos u)\, \mathrm{d}u\mathrm{d}\rho \\
&= 8\pi^2 R \int_0^r \rho\mathrm{d}\rho \\
&= 4\pi^2 R r^2
\end{aligned}$$

39　9.1

(1)

$$\mathbf{p}=\begin{pmatrix}0\\-2\\1\end{pmatrix},\mathbf{v}=\begin{pmatrix}1\\2\\3\end{pmatrix},\mathbf{w}=\begin{pmatrix}-3\\2\\-2\end{pmatrix}$$

$$\begin{aligned}\alpha_{\mathbf{p}}&=1\cdot\mathrm{d}x\wedge\mathrm{d}y-4\cdot\mathrm{d}x\wedge\mathrm{d}z+0\cdot\mathrm{d}y\wedge\mathrm{d}z\\&=\mathrm{d}x\wedge\mathrm{d}y-4\mathrm{d}x\wedge\mathrm{d}z\\&=\begin{vmatrix}1&-3\\2&2\end{vmatrix}-4\begin{vmatrix}1&-3\\3&-2\end{vmatrix}\\&=8-4\cdot7=-20\end{aligned}$$

(2)

$$\mathbf{p}=\begin{pmatrix}2\\1\\-1\end{pmatrix},\mathbf{v}_1=\begin{pmatrix}3\\2\\-1\end{pmatrix},\mathbf{v}_2=\begin{pmatrix}-1\\6\\4\end{pmatrix},\mathbf{v}_3=\begin{pmatrix}4\\-1\\2\end{pmatrix}$$

$$\begin{aligned}\omega_{\mathbf{p}}&=e^{-2}\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\\&=e^{-2}\begin{vmatrix}3&-1&4\\2&6&-1\\-1&4&2\end{vmatrix}\end{aligned}$$

40　9.2

$$\mathbf{p}=\begin{pmatrix}1\\0\\-3\\2\end{pmatrix},\mathbf{v}_1=\begin{pmatrix}1\\2\\1\\2\end{pmatrix},\mathbf{v}_2=\begin{pmatrix}2\\6\\-1\\2\end{pmatrix},\mathbf{v}_3=\begin{pmatrix}-2\\-4\\2\\1\end{pmatrix}$$

$1\le i_1\le i_2\le i_3\le 4$ から、(i_1,i_2,i_3) が可能な組合は $(1,2,3),(1,2,4),(1,3,4),(2,3,4)$

$$\begin{aligned}\omega&=\sum_{1\le i_1\le i_2\le i_3\le 4}h_{i_1i_2i_3}\mathrm{d}x_{i_1}\wedge\mathrm{d}x_{i_2}\wedge\mathrm{d}x_{i_3}\\&=h_{1,2,3}\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3+h_{1,2,4}\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_4\\&+h_{1,3,4}\mathrm{d}x_1\wedge\mathrm{d}x_3\wedge\mathrm{d}x_4+h_{2,3,4}\mathrm{d}x_2\wedge\mathrm{d}x_3\wedge\mathrm{d}x_4\\&=-2\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3+3\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_4\\&+0\mathrm{d}x_1\wedge\mathrm{d}x_3\wedge\mathrm{d}x_4-\mathrm{d}x_2\wedge\mathrm{d}x_3\wedge\mathrm{d}x_4\\&=-2\begin{vmatrix}1&2&-2\\2&6&-4\\1&-1&2\end{vmatrix}+3\begin{vmatrix}1&2&-2\\2&6&-4\\2&2&1\end{vmatrix}-\begin{vmatrix}2&6&-4\\1&-1&2\\2&2&1\end{vmatrix}\\&=-2\cdot8+3\cdot10-(-8)\\&=22\end{aligned}$$

41 9.3

$$
\begin{aligned}
f &= 3e^x \\
\alpha &= z^2 \mathrm{d}x + 2\mathrm{d}y \\
\beta &= yz\mathrm{d}x + xz\mathrm{d}y + xy\mathrm{d}z \\
\omega &= z\mathrm{d}x \wedge \mathrm{d}y + y\mathrm{d}x \wedge \mathrm{d}z + x\mathrm{d}y \wedge \mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
f \wedge \alpha &= 3e^x \left(z^2 \mathrm{d}x + 2\mathrm{d}y\right) \\
&= 3e^x z^2 \mathrm{d}x + 6e^x \mathrm{d}y
\end{aligned}
$$

$$
\alpha \wedge \alpha = 0
$$

$$
\begin{aligned}
\alpha \wedge \beta &= \left(z^2 \mathrm{d}x + 2\mathrm{d}y\right) \wedge \left(yz\mathrm{d}x + xz\mathrm{d}y + xy\mathrm{d}z\right) \\
&= xz^3 \mathrm{d}x \wedge \mathrm{d}y + xyz^2 \mathrm{d}x \wedge \mathrm{d}z + 2yz\mathrm{d}y \wedge \mathrm{d}x + 2xy\mathrm{d}y \wedge \mathrm{d}z \\
&= \left(xz^3 - 2yz\right) \mathrm{d}x \wedge \mathrm{d}y + xyz^2 \mathrm{d}x \wedge \mathrm{d}z + 2xy\mathrm{d}y \wedge \mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
\beta \wedge \omega &= \left(yz\mathrm{d}x + xz\mathrm{d}y + xy\mathrm{d}z\right) \wedge \left(z\mathrm{d}x \wedge \mathrm{d}y + y\mathrm{d}x \wedge \mathrm{d}z + x\mathrm{d}y \wedge \mathrm{d}z\right) \\
&= xyz\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + xyz\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z + xyz\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= xyz\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z
\end{aligned}
$$

42 9.4

$\omega = \sum_{1 \le i_1 < i_2 \le 4} e^{x_{i_1} + x_{i_2}} \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} + \sum_{1 \le i_2 < i_1 \le 4} e^{x_{i_1} + x_{i_2}} \mathrm{d}x_{i_2} \wedge \mathrm{d}x_{i_1} = 0$

$\omega \wedge \omega = 0 \wedge 0 = 0$

43 9.1

$$
\begin{aligned}
\omega &= \sum_{1 \le i_1 < \cdots < i_k \le n} h_{i_1 \cdots i_k} \left(\mathbf{p}\right) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \left(\mathbf{v}_1, \cdots, \mathbf{v}_k\right) \\
&= \sum_{1 \le i_1 < \cdots < i_k \le n} k \begin{vmatrix} \mathrm{d}x_{i_1}\left(\mathbf{v}_1\right) & \cdots & \mathrm{d}x_{i_1}\left(\mathbf{v}_k\right) \\ \vdots & \ddots & \vdots \\ \mathrm{d}x_{i_k}\left(\mathbf{v}_1\right) & \cdots & \mathrm{d}x_{i_k}\left(\mathbf{v}_k\right) \end{vmatrix} \\
&= \sum_{1 \le i_1 < \cdots < i_k \le n} k \begin{vmatrix} \delta_{1,1} & \cdots & \delta_{k,1} \\ \vdots & \ddots & \vdots \\ \delta_{1,k} & \cdots & \delta_{k,k} \end{vmatrix} \\
&= \sum_{1 \le i_1 < \cdots < i_k \le n} k \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \\
&= \sum_{1 \le i_1 < \cdots < i_k \le n} k \\
&= k
\end{aligned}
$$

44 9.2

$$
\begin{aligned}
f &= 2e^{-yz} \\
\alpha &= xy\mathrm{d}y + e^x\mathrm{d}z \\
\beta &= y\mathrm{d}x + \mathrm{d}y - \sin x\mathrm{d}z \\
\omega &= x\sin x\mathrm{d}x\wedge\mathrm{d}y + \sin y\mathrm{d}x\wedge\mathrm{d}z + x\cos y\mathrm{d}y\wedge\mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
f\wedge\alpha &= 2xye^{-yz}\mathrm{d}y + 2e^{x-yz}\mathrm{d}z \\
\alpha\wedge\beta &= (xy\mathrm{d}y + e^x\mathrm{d}z)\wedge(y\mathrm{d}x + \mathrm{d}y - \sin x\mathrm{d}z) \\
&= xy^2\mathrm{d}y\wedge\mathrm{d}x - xy\sin x\mathrm{d}y\wedge\mathrm{d}z + ye^x\mathrm{d}z\wedge\mathrm{d}x + e^x\mathrm{d}z\wedge\mathrm{d}y \\
\beta\wedge\omega &= (y\mathrm{d}x + \mathrm{d}y - \sin x\mathrm{d}z)\wedge(x\sin x\mathrm{d}x\wedge\mathrm{d}y + \sin y\mathrm{d}x\wedge\mathrm{d}z + x\cos y\mathrm{d}y\wedge\mathrm{d}z) \\
&= xy\cos y\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z + \sin y\mathrm{d}y\wedge\mathrm{d}x\wedge\mathrm{d}z - x\sin^2 x\mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}y \\
&= \left(xy\cos y - \sin y - x\sin^2 x\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \\
\omega\wedge\omega &= 0
\end{aligned}
$$

45 9.3

$$
\begin{aligned}
\omega &= \sum_{1\le j_1<j_2\le 4} g_{j_1j_2}\mathrm{d}x_{j_1}\wedge\mathrm{d}x_{j_2} + \sum_{1\le j_2<j_1\le 4} g_{j_2j_1}\mathrm{d}x_{j_2}\wedge\mathrm{d}x_{j_1} \\
&= \sum_{1\le j_1<j_2\le 4} x_{j_1}x_{j_2}\mathrm{d}x_{j_1}\wedge\mathrm{d}x_{j_2} + \sum_{1\le j_2<j_1\le 4} x_{j_2}x_{j_1}\mathrm{d}x_{j_2}\wedge\mathrm{d}x_{j_1} \\
&= 0
\end{aligned}
$$

から

$$
\begin{aligned}
\alpha\wedge\omega &= \alpha\wedge 0 \\
&= 0
\end{aligned}
$$

46 10.1

$$
\begin{aligned}
\mathrm{d}\alpha &= \mathrm{d}\left(ze^{x+y}\mathrm{d}x\right) \\
&= \mathrm{d}\left(ze^{x+y}\right)\wedge\mathrm{d}x \\
&= \left(ze^{x+y}\mathrm{d}x + ze^{x+y}\mathrm{d}y + e^{x+y}\mathrm{d}z\right)\wedge\mathrm{d}x \\
&= -ze^{x+y}\mathrm{d}x\wedge\mathrm{d}y - e^{x+y}\mathrm{d}x\wedge\mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}\left(2z\left(y^2-x^2\right)\mathrm{d}x\wedge\mathrm{d}y + \left(y^3-z^3\right)\mathrm{d}x\wedge\mathrm{d}z + x^3\mathrm{d}y\wedge\mathrm{d}z\right) \\
&= \mathrm{d}\left(\begin{pmatrix} x^3 \\ z^3-y^3 \\ 2z\left(y^2-x^2\right)\end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}y\wedge\mathrm{d}z \\ \mathrm{d}z\wedge\mathrm{d}x \\ \mathrm{d}x\wedge\mathrm{d}y\end{pmatrix}\right) \\
&= \left(\nabla\cdot\begin{pmatrix} x^3 \\ z^3-y^3 \\ 2z\left(y^2-x^2\right)\end{pmatrix}\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \\
&= \left(3x^2 - 3y^2 + 2\left(y^2-x^2\right)\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \\
&= \left(x^2-y^2\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z
\end{aligned}
$$

47　10.2

(1)

$$
\begin{aligned}
\mathrm{d}\alpha &= \mathrm{d}\left(\begin{pmatrix} axyz \\ bx^2z \\ -3x^2y \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix}\right) \\
&= \left(\nabla \times \begin{pmatrix} axyz \\ bx^2z \\ -3x^2y \end{pmatrix}\right) \cdot \begin{pmatrix} \mathrm{d}y \wedge \mathrm{d}z \\ \mathrm{d}z \wedge \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix} \\
&= \begin{pmatrix} -3x^2 - bx^2 \\ axy + 6xy \\ 2bxz - axz \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}y \wedge \mathrm{d}z \\ \mathrm{d}z \wedge \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix} \\
&= 0
\end{aligned}
$$

よって、$\begin{cases} -3x^2 - bx^2 = 0 \\ axy + 6xy = 0 \\ 2bxz - axz = 0 \end{cases} \implies \begin{cases} a = -6 \\ b = -3 \end{cases}$

(2)

$$
\begin{aligned}
\mathrm{d}\beta &= \mathrm{d}\left(\begin{pmatrix} -(2xe^y - ce^x) \\ (5ye^z - be^y) \\ (3ze^x + ae^z) \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}y \wedge \mathrm{d}z \\ \mathrm{d}z \wedge \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix}\right) \\
&= \left(\nabla \cdot \begin{pmatrix} -(2xe^y - ce^x) \\ (5ye^z - be^y) \\ (3ze^x + ae^z) \end{pmatrix}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= \left(-2e^y + ce^x + 5e^z - be^y + 3e^x + ae^z\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z
\end{aligned}
$$

$$
\begin{cases} (c+3)\, e^x = 0 \\ (-2-b)\, e^y = 0 \\ (5+a)\, e^z = 0 \end{cases} \implies \begin{cases} a = -5 \\ b = -2 \\ c = -3 \end{cases}
$$

48　10.3

(1)

$$
\begin{aligned}
\mathrm{d}\alpha &= \mathrm{d}\left(-\frac{y}{\sqrt{x^2+y^2}}\mathrm{d}x + \frac{x}{\sqrt{x^2+y^2}}\mathrm{d}y\right) \\
&= \mathrm{d}\left(-\frac{y}{\sqrt{x^2+y^2}}\right)\wedge \mathrm{d}x + \mathrm{d}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\wedge \mathrm{d}y \\
&= \left(\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}\mathrm{d}x - \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}\mathrm{d}y\right)\wedge \mathrm{d}x + \left(\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}\mathrm{d}x - \frac{xy}{(x^2+y^2)^{\frac{3}{2}}}\mathrm{d}y\right)\wedge \mathrm{d}y \\
&= \left(\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} + \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}\right)\mathrm{d}x\wedge \mathrm{d}y \\
&= \frac{1}{\sqrt{x^2+y^2}}\mathrm{d}x\wedge \mathrm{d}y
\end{aligned}
$$

(2)

$\alpha = \mathrm{d}f$ をみたす 0− 形式 f が存在すると、$\mathrm{d}\alpha = \mathrm{d}\,(\mathrm{d}f) = 0$
$\frac{1}{\sqrt{x^2+y^2}} \neq 0$ だから、0− 形式 f は存在しない

49　10.4

(1)

$$
\begin{aligned}
\mathrm{d}\alpha &= \mathrm{d}x_1\wedge \mathrm{d}x_3 + \mathrm{d}x_2\wedge \mathrm{d}x_4 + \mathrm{d}x_1\wedge \mathrm{d}x_3 + \mathrm{d}x_2\wedge \mathrm{d}x_4 \\
&= 2\mathrm{d}x_1\wedge \mathrm{d}x_3 + 2\mathrm{d}x_2\wedge \mathrm{d}x_4
\end{aligned}
$$

(2)

$\mathrm{d}\alpha = \mathrm{d}f\wedge \mathrm{d}g$ で、$\mathrm{d}\alpha = 2\mathrm{d}x_1\wedge \mathrm{d}x_3 + 2\mathrm{d}x_2\wedge \mathrm{d}x_4$ だから
条件をみたす 0− 形式 f, g は存在しない

50　10.1

(1)

$$
\begin{aligned}
\mathrm{d}\,(e^x\cos y\mathrm{d}x - e^x\sin y\mathrm{d}y) &= (e^x\cos y\mathrm{d}x - e^x\sin y\mathrm{d}y)\wedge \mathrm{d}x - (e^x\sin y\mathrm{d}x + e^x\cos y\mathrm{d}y)\wedge \mathrm{d}y \\
&= e^x\sin y\mathrm{d}x\wedge \mathrm{d}y - e^x\sin y\mathrm{d}x\wedge \mathrm{d}y = 0
\end{aligned}
$$

(2)

$$\begin{aligned}
\mathrm{d}\left(13x\mathrm{d}x + y^2\mathrm{d}y + xyz\mathrm{d}z\right) &= \left(\nabla \times \begin{pmatrix} 13x \\ y^2 \\ xyz \end{pmatrix}\right) \cdot \begin{pmatrix} \mathrm{d}y \wedge \mathrm{d}z \\ \mathrm{d}z \wedge \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix} \\
&= \begin{pmatrix} xz \\ -yz \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}y \wedge \mathrm{d}z \\ \mathrm{d}z \wedge \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix} \\
&= xz\mathrm{d}y \wedge \mathrm{d}z - yz\mathrm{d}z \wedge \mathrm{d}x
\end{aligned}$$

(3)

$$\begin{aligned}
\mathrm{d}\left(\mathrm{d}x \wedge \mathrm{d}y\right) &= \mathrm{d}\left(\mathrm{d}x\right) \wedge \mathrm{d}y - \mathrm{d}x \wedge \mathrm{d}\left(\mathrm{d}y\right) \\
&= 0 \wedge \mathrm{d}y - \mathrm{d}x \wedge 0 \\
&= 0
\end{aligned}$$

(4)

$$\begin{aligned}
\mathrm{d}\left(z^2\mathrm{d}x \wedge \mathrm{d}y + \left(z^2 + 2y\right)\mathrm{d}x \wedge \mathrm{d}z\right) &= \mathrm{d}\left(\begin{pmatrix} 0 \\ -\left(z^2 + 2y\right) \\ z^2 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}y \wedge \mathrm{d}z \\ \mathrm{d}z \wedge \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}y \end{pmatrix}\right) \\
&= \left(\nabla \cdot \begin{pmatrix} 0 \\ -\left(z^2 + 2y\right) \\ z^2 \end{pmatrix}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= (2z - 2)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z
\end{aligned}$$

(5)

$$\begin{aligned}
\mathrm{d}\left(z\mathrm{d}x \wedge \mathrm{d}y + y\mathrm{d}x \wedge \mathrm{d}z + x\mathrm{d}y \wedge \mathrm{d}z\right) &= \left(\nabla \cdot \begin{pmatrix} x \\ -y \\ z \end{pmatrix}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z
\end{aligned}$$

51　10.2

(1)

$$\begin{aligned}
\mathrm{d}\alpha &= -\mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}x_3 \wedge \mathrm{d}x_4 - \mathrm{d}x_3 \wedge \mathrm{d}x_4 \\
&= -2\mathrm{d}x_1 \wedge \mathrm{d}x_2 - 2\mathrm{d}x_3 \wedge \mathrm{d}x_4
\end{aligned}$$

(2)

$\mathrm{d}\alpha = \theta \wedge \phi$ をみたす θ, ϕ が存在すると仮定すると、$\mathrm{d}\alpha \wedge \mathrm{d}\alpha = 0$ より

$$\begin{aligned}
0 &= (-2\mathrm{d}x_1 \wedge \mathrm{d}x_2 - 2\mathrm{d}x_3 \wedge \mathrm{d}x_4) \wedge (-2\mathrm{d}x_1 \wedge \mathrm{d}x_2 - 2\mathrm{d}x_3 \wedge \mathrm{d}x_4) \\
&= 4\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 + 4\mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \\
&= 8\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \\
&\neq 0
\end{aligned}$$

矛盾するから、条件を満たす θ, ϕ は存在しない

52　11.1

$$\omega = \left(x^2+y^2\right) \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}z - \mathrm{d}y \wedge \mathrm{d}z$$
$$\phi = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$$

(1)

$$\phi^*\omega = \left(u^2+v^2\right) \mathrm{d}u \wedge \mathrm{d}v + \mathrm{d}u \wedge \mathrm{d}0 - \mathrm{d}v \wedge \mathrm{d}0 \tag{46}$$
$$= \left(u^2+v^2\right) \mathrm{d}u \wedge \mathrm{d}v \tag{47}$$

(2)

$$\int_{\phi|K} \omega = \iint_{\{u^2+v^2<1\}} \mathrm{d}u\mathrm{d}v \tag{48}$$
$$= \int_0^1 \int_0^{2\pi} r^2 \cdot r\mathrm{d}\theta\mathrm{d}\rho \tag{49}$$
$$= 2\pi \int_0^1 r^3 \mathrm{d}r \tag{50}$$
$$= \frac{\pi}{2} \tag{51}$$

53　11.2

$$\omega = \frac{z}{\sqrt{x^2+y^2+z^2}}\mathrm{d}x \wedge \mathrm{d}y - \frac{y}{\sqrt{x^2+y^2+z^2}}\mathrm{d}x \wedge \mathrm{d}z + \frac{x}{\sqrt{x^2+y^2+z^2}}\mathrm{d}y \wedge \mathrm{d}z$$
$$\phi(u,v) = \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix}$$

(1)

$$\phi^*\omega = z\mathrm{d}x \wedge \mathrm{d}y + y\mathrm{d}z \wedge \mathrm{d}x + x\mathrm{d}y \wedge \mathrm{d}z \tag{52}$$

ここで $\begin{cases} \mathrm{d}x = \cos u \cos v \mathrm{d}u - \sin u \sin v \mathrm{d}v \\ \mathrm{d}y = \cos u \sin v \mathrm{d}u + \sin u \cos v \mathrm{d}v \\ \mathrm{d}z = -\sin u \mathrm{d}u \end{cases}$

$$\begin{aligned} \mathrm{d}x \wedge \mathrm{d}y &= (\cos u \cos v \mathrm{d}u - \sin u \sin v \mathrm{d}v) \wedge (\cos u \sin v \mathrm{d}u + \sin u \cos v \mathrm{d}v) \\ &= \sin u \cos u \cos^2 v \mathrm{d}u \wedge \mathrm{d}v - \sin u \cos u \sin^2 v \mathrm{d}v \wedge \mathrm{d}u \\ &= \sin u \cos u \mathrm{d}u \wedge \mathrm{d}v \\ \mathrm{d}y \wedge \mathrm{d}z &= (\cos u \sin v \mathrm{d}u + \sin u \cos v \mathrm{d}v) \wedge (-\sin u \mathrm{d}u) \\ &= \sin^2 u \cos v \mathrm{d}u \wedge \mathrm{d}v \end{aligned}$$

$$\begin{aligned}\mathrm{d}z\wedge\mathrm{d}x &= (-\sin u\mathrm{d}u)\wedge(\cos u\cos v\mathrm{d}u-\sin u\sin v\mathrm{d}v)\\ &= \sin^2 u\sin v\mathrm{d}u\wedge\mathrm{d}v\end{aligned}$$

$$\begin{aligned}\phi^*\omega &= z\mathrm{d}x\wedge\mathrm{d}y+y\mathrm{d}z\wedge\mathrm{d}x+x\mathrm{d}y\wedge\mathrm{d}z && (53)\\ &= \sin u\cos^2 u\mathrm{d}u\wedge\mathrm{d}v+\sin^3 u\sin^2 v\mathrm{d}u\wedge\mathrm{d}v+\sin^3 u\cos^2 v\mathrm{d}u\wedge\mathrm{d}v && (54)\\ &= \sin u\left(\cos^2 u+\sin^2 u\sin^2 v+\sin^2 u\cos^2 v\right)\mathrm{d}u\wedge\mathrm{d}v && (55)\\ &= \sin u\mathrm{d}u\wedge\mathrm{d}v && (56)\end{aligned}$$

(2)

$$\begin{aligned}\int_{\phi|K}\omega &= \iint_{(0,\pi)\times(0,2\pi)}\sin u\mathrm{d}u\mathrm{d}v && (57)\\ &= \int_0^{2\pi}\int_0^{\pi}\sin u\mathrm{d}u\mathrm{d}v && (58)\\ &= 4\pi && (59)\end{aligned}$$

54　11.1

(1)

$\mathrm{d}z=2u\mathrm{d}u-2v\mathrm{d}v$ だから、$\begin{cases}\mathrm{d}z\wedge\mathrm{d}x=(2u\mathrm{d}u-2v\mathrm{d}v)\wedge\mathrm{d}u=2v\mathrm{d}u\wedge\mathrm{d}v\\ \mathrm{d}y\wedge\mathrm{d}z=\mathrm{d}v\wedge(2u\mathrm{d}u-2v\mathrm{d}v)=-2u\mathrm{d}u\wedge\mathrm{d}v\end{cases}$

$$\begin{aligned}\phi^*\omega &= -\left(v^2+uv\right)\mathrm{d}u\wedge\mathrm{d}v-2v\left(u-u^2v+v^3\right)\mathrm{d}u\wedge\mathrm{d}v-2u\left(v+u^3-uv^2\right)\mathrm{d}u\wedge\mathrm{d}v && (60)\\ &= \left(-v^2-uv-2uv+2u^2v^2-2v^4-2uv-2u^4+2u^2v^2\right)\mathrm{d}u\wedge\mathrm{d}v && (61)\\ &= \left(-2u^4-5uv-v^2+4u^2v^2-2v^4\right)\mathrm{d}u\wedge\mathrm{d}v && (62)\end{aligned}$$

(2)

$$\begin{aligned}\int_{\phi|K}\omega &= \iint_{\{u^2+v^2<9\}}\left(-2u^4-5uv-v^2+4u^2v^2-2v^4\right)\mathrm{d}u\mathrm{d}v && (63)\\ &= \int_{-3}^{3}\int_{-\sqrt{9-u^2}}^{\sqrt{9-u^2}}\left(-2u^4-5uv-v^2+4u^2v^2-2v^4\right)\mathrm{d}v\mathrm{d}u && (64)\\ &= -\frac{2}{15}\int_{-3}^{3}\sqrt{9-u^2}\left(531-293u^2+56u^4\right)\mathrm{d}u && (65)\\ &= -\frac{1053}{4}\pi && (66)\end{aligned}$$

55　11.2

(1)

ここで $\begin{cases} \mathrm{d}x = \cos u \cos v \mathrm{d}u - \sin u \sin v \mathrm{d}v \\ \mathrm{d}y = \cos u \sin v \mathrm{d}u + \sin u \cos v \mathrm{d}v \\ \mathrm{d}z = -\sin u \mathrm{d}u \end{cases}$

$$\begin{aligned} \mathrm{d}x \wedge \mathrm{d}y &= (\cos u \cos v \mathrm{d}u - \sin u \sin v \mathrm{d}v) \wedge (\cos u \sin v \mathrm{d}u + \sin u \cos v \mathrm{d}v) \\ &= \sin u \cos u \cos^2 v \mathrm{d}u \wedge \mathrm{d}v - \sin u \cos u \sin^2 v \mathrm{d}v \wedge \mathrm{d}u \\ &= \sin u \cos u \mathrm{d}u \wedge \mathrm{d}v \\ \mathrm{d}y \wedge \mathrm{d}z &= (\cos u \sin v \mathrm{d}u + \sin u \cos v \mathrm{d}v) \wedge (-\sin u \mathrm{d}u) \\ &= \sin^2 u \cos v \mathrm{d}u \wedge \mathrm{d}v \end{aligned}$$

$$\begin{aligned} \mathrm{d}z \wedge \mathrm{d}x &= (-\sin u \mathrm{d}u) \wedge (\cos u \cos v \mathrm{d}u - \sin u \sin v \mathrm{d}v) \\ &= \sin^2 u \sin v \mathrm{d}u \wedge \mathrm{d}v \end{aligned}$$

$$\phi^* \omega = z^3 \mathrm{d}x \wedge \mathrm{d}y + y^3 \mathrm{d}z \wedge \mathrm{d}x + x^3 \mathrm{d}y \wedge \mathrm{d}z \tag{67}$$

$$= \left(\sin u \cos^4 u + \sin^5 u \sin^4 v + \sin^5 u \cos^4 v \right) \mathrm{d}u \wedge \mathrm{d}v \tag{68}$$

$$= \sin u \left(\frac{1}{4} \sin^4 u (\cos 4v + 3) + \cos^4 u \right) \mathrm{d}u \wedge \mathrm{d}v \tag{69}$$

(2)

$$\int_{\phi|K} \omega = \iint_{(0,\pi)\times(0,2\pi)} \sin u \left(\frac{1}{4} \sin^4 u (\cos 4v + 3) + \cos^4 u \right) \mathrm{d}u \mathrm{d}v \tag{70}$$

$$= \int_0^\pi \frac{1}{4} \sin^5 u \int_0^{2\pi} (\cos 4v + 3) \mathrm{d}v \mathrm{d}u + \int_0^\pi \int_0^{2\pi} \sin u \cos^4 u \mathrm{d}v \mathrm{d}u \tag{71}$$

$$= \frac{3}{2}\pi \int_0^\pi \sin^5 u \mathrm{d}u + 2\pi \int_0^\pi \sin u \cos^4 u \mathrm{d}u \tag{72}$$

$$= \frac{8}{5}\pi + \frac{4}{5}\pi \tag{73}$$

$$= \frac{12}{5}\pi \tag{74}$$

56　12.1

$$\begin{aligned} \alpha(\gamma(t)) &= \frac{rt \cos t}{r} \mathrm{d}(r \cos t) + \frac{rt \sin t}{r} \mathrm{d}(r \sin t) + r \mathrm{d}t \\ &= t \cos t (-r \sin t \mathrm{d}t) + t \sin t (r \cos t \mathrm{d}t) + r \mathrm{d}t \\ &= -rt \sin t \cos t \mathrm{d}t + rt \sin t \cos t \mathrm{d}t + r \mathrm{d}t \\ &= r \mathrm{d}t \end{aligned}$$

$$\begin{aligned} \int_\gamma \alpha &= \int_0^{2\pi} r \mathrm{d}t \\ &= 2\pi r \end{aligned}$$

57 12.2

$$
\begin{aligned}
\phi^*\omega &= -u^2v\mathrm{d}u + u^3\mathrm{d}v + u^2\left(u^2+v^2\right)\mathrm{d}\left(u^2+v^2\right) \\
&= -u^2v\mathrm{d}u + u^3\mathrm{d}v + u^2\left(u^2+v^2\right)\left(2u\mathrm{d}u + 2v\mathrm{d}v\right) \\
&= -u^2v\mathrm{d}u + u^3\mathrm{d}v + 2u^3\left(u^2+v^2\right)\mathrm{d}u + 2u^2v\left(u^2+v^2\right)\mathrm{d}v \\
&= \left(2u^5 + 2u^3v^2 - u^2v\right)\mathrm{d}u + \left(2u^4v + 2u^2v^3 + u^3\right)\mathrm{d}v
\end{aligned}
$$

$$
\int_{\phi|\overline{\Omega}} \mathrm{d}\omega = \int_{\{u^2+v^2=4\}} \left(\left(2u^5 + 2u^3v^2 - u^2v\right)\mathrm{d}u + \left(2u^4v + 2u^2v^3 + u^3\right)\mathrm{d}v\right)
$$

$$
\begin{cases}
\left(2u^5 + 2u^3v^2 - u^2v\right)\mathrm{d}u = \left(64\cos^5 t + 64\sin^2 t\cos^3 t - 8\sin t\cos^2 t\right)\left(-2\sin t\mathrm{d}t\right) \\
\left(2u^4v + 2u^2v^3 + u^3\right)\mathrm{d}v = \left(64\sin t\cos^4 t + 64\sin^3 t\cos^2 t + 8\cos^3 t\right)\left(2\cos t\mathrm{d}t\right)
\end{cases}
$$

$$
\begin{cases}
\left(2u^5 + 2u^3v^2 - u^2v\right)\mathrm{d}u = -16\left(8\sin t\cos^5 t + 8\sin^3 t\cos^3 t - \sin^2 t\cos^2 t\right)\mathrm{d}t \\
\left(2u^4v + 2u^2v^3 + u^3\right)\mathrm{d}v = 16\left(8\sin t\cos^5 t + 8\sin^3 t\cos^3 t + \cos^4 t\right)\mathrm{d}t
\end{cases}
$$

$$
\Longrightarrow \left(2u^5 + 2u^3v^2 - u^2v\right)\mathrm{d}u + \left(2u^4v + 2u^2v^3 + u^3\right)\mathrm{d}v = 16\sin^2 t\cos^2 t + 16\cos^4 t
$$

$$
\begin{aligned}
\int_{\phi|\overline{\Omega}} \mathrm{d}\omega &= \int_0^{2\pi} \left(16\sin^2 t\cos^2 t + 16\cos^4 t\right)\mathrm{d}t \\
&= \int_0^{2\pi} \cos^2 t\mathrm{d}t \\
&= \pi
\end{aligned}
$$

58 12.3

$$
\int_S (\alpha + \mathrm{d}f)\wedge(\beta + \mathrm{d}g) = \int_S (\alpha\wedge\beta + \mathrm{d}f\wedge\beta + \alpha\wedge\mathrm{d}g + \mathrm{d}f\wedge\mathrm{d}g)
$$

$$
\begin{cases}
\mathrm{d}\left(\alpha g\right) = \mathrm{d}\alpha\wedge g - \alpha\wedge\mathrm{d}g = -\alpha\wedge\mathrm{d}g \\
\mathrm{d}\left(\beta f\right) = \mathrm{d}\beta\wedge f - \beta\wedge\mathrm{d}f = -\beta\wedge\mathrm{d}f \\
\mathrm{d}\left(f\mathrm{d}g\right) = \mathrm{d}f\wedge\mathrm{d}g - f\wedge\mathrm{d}\left(\mathrm{d}g\right) = \mathrm{d}f\wedge\mathrm{d}g
\end{cases}
$$

$$
\begin{aligned}
\int_S (\alpha + \mathrm{d}f)\wedge(\beta + \mathrm{d}g) &= \int_S \left(\alpha\wedge\beta + \mathrm{d}\left(\beta f\right) - \mathrm{d}\left(\alpha g\right) + \mathrm{d}\left(f\mathrm{d}g\right)\right) \\
&= \int_S \left(\alpha\wedge\beta + \left(\nabla\times\left(\beta f\right)\right) - \left(\nabla\times\left(\alpha g\right)\right) + \left(\nabla\times\left(f\mathrm{d}g\right)\right)\right) \\
&= \int_S \alpha\wedge\beta
\end{aligned}
$$

59 12.4

$$
\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}\left(\begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2+z^2}} \\ \dfrac{y}{\sqrt{x^2+y^2+z^2}} \\ \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}y\wedge\mathrm{d}z \\ \mathrm{d}z\wedge\mathrm{d}x \\ \mathrm{d}x\wedge\mathrm{d}y \end{pmatrix}\right) \\
&= \frac{2}{\sqrt{x^2+y^2+z^2}}\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
\int_{\partial V} \omega &= \int_{\overline{V}} \mathrm{d}\omega \\
&= \int_0^{2r}\int_0^{\pi}\int_0^{2\pi} \frac{2}{\rho}\cdot\rho^2\sin\phi \mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\rho - \int_0^{r}\int_0^{\pi}\int_0^{2\pi} \frac{2}{\rho}\cdot\rho^2\sin\phi \mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\rho \\
&= 2\pi\left(\int_0^{2r}\int_0^{\pi} 2\rho\sin\phi\mathrm{d}\phi\mathrm{d}\rho - \int_0^{r}\int_0^{\pi} 2\rho\sin\phi\mathrm{d}\phi\mathrm{d}\rho\right) \\
&= 8\pi\left(\int_0^{2r}\rho\mathrm{d}\rho - \int_0^{r}\rho\mathrm{d}\rho\right) \\
&= 12\pi r^2
\end{aligned}
$$

60　12.1

$\alpha = \mathrm{d}f$ とすると、$\begin{cases} \dfrac{\partial f}{\partial x} = x\left(r^2 - z^2\right) \\ \dfrac{\partial f}{\partial y} = y\left(r^2 - z^2\right) \\ \dfrac{\partial f}{\partial z} = -z\left(x^2 + y^2\right) \end{cases} \Longrightarrow f = \displaystyle\int x\left(r^2 - z^2\right)\mathrm{d}x = \frac{1}{2}x^2\left(r^2 - z^2\right) + g\left(y, z\right)$

これを $\dfrac{\partial f}{\partial y} = y\left(r^2 - z^2\right)$ に代入すると $\dfrac{\partial}{\partial y}\left[\dfrac{1}{2}x^2\left(r^2 - z^2\right) + g\left(y, z\right)\right] = y\left(r^2 - z^2\right)$

$\iff \dfrac{\partial}{\partial y}g\left(y, z\right) = y\left(r^2 - z^2\right) \iff g\left(y, z\right) = \displaystyle\int y\left(r^2 - z^2\right)\mathrm{d}y = \frac{1}{2}y^2\left(r^2 - z^2\right) + h\left(z\right)$

$\dfrac{\partial f}{\partial z} = \dfrac{\partial}{\partial z}\left[\dfrac{1}{2}x^2\left(r^2 - z^2\right) + \dfrac{1}{2}y^2\left(r^2 - z^2\right) + h\left(z\right)\right] = -\left(x^2 + y^2\right)z + \dfrac{\partial}{\partial z}h\left(z\right) = -z\left(x^2 + y^2\right)$

$\Longrightarrow h\left(z\right) = 0$

以上より、$\alpha = \mathrm{d}f$ をみたす $f = \dfrac{1}{2}\left(x^2 + y^2\right)\left(r^2 - z^2\right)$

$$
\begin{aligned}
\int_\gamma \alpha &= f\left(\gamma\left(2\pi\right)\right) - f\left(\gamma\left(0\right)\right) \\
&= f\left(R + r, 0, 0\right) - f\left(R + r, 0, 0\right) = 0
\end{aligned}
$$

61　12.2

$$
\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}\left(\begin{pmatrix} x\sin\frac{z}{k} - y\cos\frac{z}{k} \\ x\cos\frac{z}{k} + y\sin\frac{z}{k} \\ x^2 + y^2 + \frac{z^2}{k^2} \end{pmatrix}\cdot\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix}\right) \\
&= \begin{pmatrix} 2y + \dfrac{x\sin\frac{z}{k}}{k} - \dfrac{y\cos\frac{z}{k}}{k} \\ -2x + \dfrac{x\cos\frac{z}{k}}{k} + \dfrac{y\sin\frac{z}{k}}{k} \\ 2\cos\dfrac{z}{k} \end{pmatrix}\cdot\begin{pmatrix} \mathrm{d}y\wedge\mathrm{d}z \\ \mathrm{d}z\wedge\mathrm{d}x \\ \mathrm{d}x\wedge\mathrm{d}y \end{pmatrix}
\end{aligned}
$$

62　13.1

$\phi\left(\mathbb{R}^2\right)=S^2\backslash\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

$\phi(u,v)=\begin{pmatrix}0\\0\\1\end{pmatrix}+S\left(\begin{pmatrix}u\\v\\0\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\in S^2(1)$ とすると $\phi(u,v)=\begin{pmatrix}su\\sv\\1-s\end{pmatrix}$ で $(su)^2+(sv)^2+(1-s)^2=1 \iff \left(u^2+v^2+1\right)s^2-2s=0$ より、$s=\dfrac{2}{u^2+v^2+1}$ となる

$\psi^j=\phi\circ\partial_j^1=\begin{pmatrix}\psi_1^j\\\psi_2^j\\\psi_3^j\end{pmatrix}$ とする

$\psi^0(t)=\begin{pmatrix}\dfrac{1-t}{t^2-t+1}\\\dfrac{t}{t^2-t+1}\\1-\dfrac{1}{t^2-t+1}\end{pmatrix},\psi^1(t)=\begin{pmatrix}0\\\dfrac{2t}{t^2+1}\\1-\dfrac{2}{t^2+1}\end{pmatrix},\psi^2(t)=\begin{pmatrix}\dfrac{2t}{t^2+1}\\0\\1-\dfrac{2}{t^2+1}\end{pmatrix}$ となる

$\mathrm{d}\psi_1^0\dfrac{t^2-2t}{(t^2-t+1)^2}\mathrm{d}t,\mathrm{d}\psi_2^0=-\dfrac{t^2-1}{(t^2-t+1)^2}\mathrm{d}t,\mathrm{d}\psi_3^0=\dfrac{2t-1}{(t^2-t+1)^2}\mathrm{d}t$ より

$\left(\psi^0\right)^*\alpha=\left(\dfrac{1-2t}{(t^2-t+1)^3}+\left(\dfrac{1}{t^2-t+1}+1\right)\dfrac{2t-1}{(t^2-t+1)^2}\right)\mathrm{d}t=\dfrac{2t-1}{(t^2-t+1)^2}\mathrm{d}t$ なので

$$\begin{aligned}\int_{\phi\circ\partial_0^1|\Delta^1}\alpha&=\int_0^1\frac{2t-1}{(t^2-t+1)^2}\mathrm{d}t\\&=\int\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{t^2-t+1}\right)\mathrm{d}t\\&=0\end{aligned}$$

また $\mathrm{d}\psi_1^1=0,\mathrm{d}\psi_2^1=\dfrac{2\left(1-t^2\right)}{(t^2+1)^2}\mathrm{d}t,\mathrm{d}\psi_3^1=\dfrac{4t}{(t^2+1)^2}\mathrm{d}t$ より、$\left(\psi^1\right)^*\alpha=\dfrac{4(t+1)}{(t^2+1)^2}\mathrm{d}t$ なので、

$\displaystyle\int_{\phi\circ\partial_1^1|\Delta^1}\omega=\int_0^1\frac{4(t+1)}{(t^2+1)^2}\mathrm{d}t=\frac{\pi}{2}$

$\mathrm{d}\psi_1^2=\dfrac{2\left(1-t^2\right)}{(t^2+1)^2}\mathrm{d}t,\mathrm{d}\psi_2^2=0,\mathrm{d}\psi_3^2=\dfrac{4t}{(t^2+1)^2}\mathrm{d}t$ より $\left(\psi^2\right)^*\alpha=\dfrac{8t(t+1)}{(t^2+1)^3}\mathrm{d}t$ より

$$\begin{aligned}\int_{\phi\circ\partial_2^1|\Delta^1}\omega&=\int_0^1\frac{8t(t+1)}{(t^2+1)^3}\mathrm{d}t\\&=\pi+2\end{aligned}$$

以上より、$\displaystyle\int_{\phi|\Delta^2}\mathrm{d}\omega=\frac{\pi}{2}+\frac{7}{2}$

63　13.2

$\psi^j = \phi \circ \partial_j^2 = \begin{pmatrix} \psi_1^j \\ \psi_2^j \\ \psi_3^j \\ \psi_4^j \end{pmatrix}$ とする

$\psi^0(t_1, t_2) = (1 - t_1 - t_2)\begin{pmatrix} \cos t_1 \\ \sin t_2 \\ \cos t_2 \\ \sin t_2 \end{pmatrix}, \psi^1(t_1, t_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\psi^2(t_1, t_2) = \begin{pmatrix} t_1 \\ 0 \\ t_1 \cos t_2 \\ t_1 \sin t_2 \end{pmatrix}, \psi^3(t_1, t_2) = \begin{pmatrix} t_1 \cos t_2 \\ t_1 \sin t_2 \\ t_1 \\ 0 \end{pmatrix}$ より、$\left(\psi^0\right)^* \omega = 0, \left(\psi^1\right)^* \omega = 0$ より、

$\int_{\phi \circ \partial_0^2 | \Delta^2} \omega = \int_{\phi \circ \partial_1^2 | \Delta^2} \omega = 0$

$\left(\psi^2\right)^* \omega = t_1^3 \mathrm{d}t_1 \wedge \mathrm{d}t_2, \left(\psi^3\right)^* \omega = t_1^3 \mathrm{d}t_1 \wedge \mathrm{d}t_2$

よって、$\int_{\phi \circ \partial_2^2 | \Delta^2} \omega = \int_{\phi \circ \partial_3^2 | \Delta^2} \omega = \iint_{\Delta^2} t_1^3 \mathrm{d}t_1 \mathrm{d}t_2$

以上より $\int_{\phi | \Delta^3} \mathrm{d}\omega = \int_{\phi \circ \partial_0^2 | \Delta^2} \omega - \int_{\phi \circ \partial_1^2 | \Delta^2} \omega + \int_{\phi \circ \partial_2^2 | \Delta^2} \omega - \int_{\phi \circ \partial_3^2 | \Delta^2} \omega$