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1 §1

1.1 E1.1

(1)

$f=x^2+4y^2+z^2-4x-2z+1$ とすると，f は C^∞ であるから，$S\cap\mathbb{R}^3=f^{-1}\left(\{0\}\right)$

$\forall p=\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3,\left(\nabla f\right)\left(p\right)=\begin{pmatrix}2x-4\\8y\\2z-2\end{pmatrix}$ から，$\left(\nabla f\right)\left(p\right)=0$ となるのは $\begin{pmatrix}2\\0\\1\end{pmatrix}$ だけである

が，$\begin{pmatrix}2\\0\\1\end{pmatrix}\notin S$ ので，$\forall p\in S,\left(\nabla f\right)\left(p\right)\neq 0$

以上より，S が正則曲面である

(2)

$x^2+4y^2+z^2-4x-2z+1=0\iff(x-2)^2+4y^2+(z-1)^2=4\iff\left(\frac{x-2}{2}\right)^2+y^2+\left(\frac{z-1}{2}\right)^2=1$

よって，$\sigma\left(u,v\right)=\begin{pmatrix}2\sin u\cos v+2\\\sin u\sin v\\2\cos u+1\end{pmatrix}$ とすると，$D:=(0,\pi)\times(0,2\pi)$ とし，$\sigma\left(D\right)\subset S$ で

あり，$p=\sigma\left(\frac{\pi}{2},\pi\right)$ となる．また，$\sigma\left(D\right)$ は σ でパラメータ表示された曲面片であるから，σ は S の局所パラメータ表示である

(3)

$T_pS=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3\middle|\left(\nabla f\right)\left(p\right)\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}\right\}=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3\middle|\begin{pmatrix}-4\\0\\0\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}\right\}$ で

$\begin{pmatrix}-4\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}=0$ であるから，$v\in T_pS$

$\sigma_u=\begin{pmatrix}2\cos u\cos v\\\cos u\sin v\\-2\sin u\end{pmatrix},\sigma_v=\begin{pmatrix}-2\sin u\sin v\\\sin u\cos v\\0\end{pmatrix}$ だから

$\sigma_u\left(\frac{\pi}{2},\pi\right)=\begin{pmatrix}0\\0\\-2\end{pmatrix},\sigma_v\left(\frac{\pi}{2},\pi\right)=\begin{pmatrix}0\\-1\\0\end{pmatrix}\implies v=\begin{pmatrix}0\\1\\1\end{pmatrix}=-\frac{1}{2}\sigma_u\left(\frac{\pi}{2},\pi\right)-\sigma_v\left(\frac{\pi}{2},\pi\right)$

$\implies\gamma\left(t\right)=\sigma\left(\frac{\pi}{2}-\frac{1}{2}t,\pi-t\right)$

1.2 E1.2

(1)

$f=x^2+y^2-2xz-1$ とすると，f は C^∞ であるから，$S\cap\mathbb{R}^3=f^{-1}\left(\{0\}\right)$

$\forall p\in\mathbb{R}^3,\left(\nabla f\right)\left(p\right)=\begin{pmatrix}2\left(x-z\right)\\2y\\-2x\end{pmatrix},\left(\nabla f\right)\left(p\right)=0$ となるのは $\begin{pmatrix}0\\0\\0\end{pmatrix}$ だけであるが $\begin{pmatrix}0\\0\\0\end{pmatrix}\notin S$

よって $\forall p\in S,\left(\nabla f\right)\left(p\right)\neq 0$

(2)

$x^2+y^2-2xz=1 \Longleftrightarrow (x-z)^2+y^2=z^2+1$ で，中心 $(z_0,0)$，半径 $\sqrt{z^2+1}$ の円を表ている

$D:=(0,2\pi)\times\mathbb{R}$ とし，$\sigma(u,v)=\begin{pmatrix}\sqrt{v^2+1}\cos u+v\\ \sqrt{v^2+1}\sin u+v\\ v\end{pmatrix}$ とすれば，$\sigma(D)\subset S$ かつ $p=\sigma(\pi,0)$ であり，さらに $\sigma(D)$ は S でパラメータ表示された曲面片であるから，σ は S の局所パラメータ表示である

(3)

$(\nabla f)(p)\cdot v=\begin{pmatrix}-2\\0\\2\end{pmatrix}\cdot\begin{pmatrix}2\\0\\2\end{pmatrix}=0$ から $v\in T_pS$

また $v=2\sigma_v(\pi,0)$ であるから，$\gamma(t)=\sigma(\pi+0t,0+2t)=\sigma(\pi,2t)$

1.3　P1.1

(1)

$f=xy-1$ とすると，f は C^∞ であるから，$S\cap\mathbb{R}^3=f^{-1}(\{0\})$

$(\nabla f)(p)=\begin{pmatrix}y\\x\\0\end{pmatrix}=0$ をみたす p は $\begin{pmatrix}0\\0\\0\end{pmatrix}$ だけであるが

$\begin{pmatrix}0\\0\\0\end{pmatrix}\notin S$ だから，$\forall p\in S,(\nabla f)(p)\neq 0$

以上より，S が正則曲面である

(2)

$\sigma(u,v)=\begin{pmatrix}u\\ \frac{1}{u}\\ v\end{pmatrix}$ とおくと，$D:=(\mathbb{R}-\{0\})\times\mathbb{R}$ とし，$\sigma(D)\subset S$ かつ $p=\sigma(-1,1)$ であり，さらに $\sigma(D)$ は S でパラメータ表示された曲面片であるから，σ は S の局所パラメータ表示である

(3)

$(\nabla f)(p)\cdot v=\begin{pmatrix}-1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=0$ から $v\in T_pS$

$\sigma_u=\begin{pmatrix}1\\-\frac{1}{u^2}\\0\end{pmatrix},\sigma_v=\begin{pmatrix}0\\0\\1\end{pmatrix}$ で，$v=1\cdot\sigma_u(-1,1)+3\cdot\sigma_v(-1,1)$ から，$\gamma(t)=\sigma(t-1,3t+1)$

1.4　P1.2

(1)

$f=3x^2+2y^2-6z$ とすると，$f:C^\infty$ より，$S\cap\mathbb{R}^3=f^{-1}\left(\{0\}\right)$

$\left(\nabla f\right)(p)=\begin{pmatrix}6x\\4y\\-6\end{pmatrix}$ の第三成分より $\forall p\in S,\left(\nabla f\right)(p)\neq 0$

以上より，S が正則曲面

(2)

$\sigma\left(u,v\right)=\begin{pmatrix}u\\v\\\frac{1}{6}\left(3u^2+2v^2\right)\end{pmatrix}$ とおくと，$D:=\mathbb{R}^2$ とし，$\sigma\left(D\right)\subset S$ かつ $p=\sigma\left(0,\sqrt{3}\right)$ であり，

$\sigma\left(D\right)$ は S でパラメータ表示された曲面片であるから，σ は S のパラメータ表示である

(3)

$\left(\nabla f\right)(p)\cdot v=\begin{pmatrix}0\\4\sqrt{3}\\-6\end{pmatrix}\cdot\begin{pmatrix}-1\\\sqrt{3}\\2\end{pmatrix}=0$ から，$v\in T_pS$

$\sigma_u=\begin{pmatrix}1\\0\\u\end{pmatrix},\sigma_v=\begin{pmatrix}0\\1\\\frac{2}{3}v\end{pmatrix}$ で,$v=-1\cdot\sigma_u\left(0,\sqrt{3}\right)+\sqrt{3}\cdot\sigma_v\left(0,\sqrt{3}\right)$ から,$\gamma\left(t\right)=\sigma\left(-t,\sqrt{3}\left(1+t\right)\right)$

1.5　P1.3

(1)

$f=3x^2-2y^2-6z$ とすると，$f:C^\infty$ より，$S\cap\mathbb{R}^3=f^{-1}\left(\{0\}\right)$

$\left(\nabla f\right)(p)=\begin{pmatrix}6x\\-4y\\-6\end{pmatrix}$ の第三成分より $\forall p\in S,\left(\nabla f\right)(p)\neq 0$

以上より，S が正則曲面

(2)

$\sigma\left(u,v\right)=\begin{pmatrix}u\\v\\\frac{1}{6}\left(3u^2-2v^2\right)\end{pmatrix}$ とおくと，$D:=\mathbb{R}^2$ とし，$\sigma\left(D\right)\subset S$ かつ $p=\sigma\left(2,-3\right)$ であり，

$\sigma\left(D\right)$ は S でパラメータ表示された曲面片であるから，σ は S のパラメータ表示である

(3)

$\sigma_u=\begin{pmatrix}1\\0\\u\end{pmatrix},\sigma_v=\begin{pmatrix}0\\1\\-\frac{2}{3}v\end{pmatrix}$ で,$v=2\cdot\sigma_u\left(2,-3\right)-3\cdot\sigma_v\left(2,-3\right)$ から,$\gamma\left(t\right)=\sigma\left(2+2t,-3-3t\right)$

1.6 P1.4

(1)

$f=x^2+y^2-z^2-1$ とすると，$f:C^\infty$ より $S\cap\mathbb{R}^3=f^{-1}(\{0\})$
$(\nabla f)(p)=\begin{pmatrix}2x\\2y\\-2z\end{pmatrix}=0$ となるのは $\begin{pmatrix}0\\0\\0\end{pmatrix}$ だけであるから，$\begin{pmatrix}0\\0\\0\end{pmatrix}\notin S$ から，$\forall p\in S,(\nabla f)(p)\neq 0$
以上より，S が正則曲面である

(2)

$\sigma(u,v)=\begin{pmatrix}\sqrt{v^2+1}\cos u\\\sqrt{v^2+1}\sin u\\v\end{pmatrix}$ とき，$D:(0,2\pi)\times\mathbb{R}$ とし，$\sigma(D)\subset S$ かつ $p=\sigma\left(\frac{\pi}{4},-1\right)$ であり，$\sigma(D)$ は S でパラメータ表示された曲面片であるから，σ は S のパラメータ表示である

(3)

$(\nabla f)(p)\cdot v=\begin{pmatrix}2\\2\\2\end{pmatrix}\cdot\begin{pmatrix}-4\\2\\2\end{pmatrix}=0$ から，$v\in T_pS$
$\sigma_u=\begin{pmatrix}-\sqrt{v^2+1}\sin u\\\sqrt{v^2+1}\cos u\\0\end{pmatrix}\sigma_v=\begin{pmatrix}\frac{v}{\sqrt{v^2+1}}\cos u\\\frac{v}{\sqrt{v^2+1}}\sin u\\1\end{pmatrix}$
$\Longrightarrow\sigma_u\left(\frac{\pi}{4},-1\right)=\begin{pmatrix}-1\\1\\0\end{pmatrix},\sigma_v\left(\frac{\pi}{4},-1\right)=\begin{pmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{pmatrix}$
$\Longrightarrow v=3\cdot\sigma_u\left(\frac{\pi}{4},-1\right)+2\cdot\sigma_v\left(\frac{\pi}{4},-1\right)$ から，$\gamma(t)=\sigma\left(3t+\frac{\pi}{4},2t-1\right)$

1.7 P1.5

(1)

$f=x^2+y^2-z^2+1$ とすると，$f:C^\infty$ より，$S\cap\mathbb{R}^3=f^{-1}(\{0\})$
$(\nabla f)(p)=\begin{pmatrix}2x\\2y\\-2z\end{pmatrix}=0$ となるのは $\begin{pmatrix}0\\0\\0\end{pmatrix}$ だけであり，$\begin{pmatrix}0\\0\\0\end{pmatrix}\notin S$ から，$\forall p\in S,(\nabla f)(p)\neq 0$
以上より，S が正則曲面である

(2)

$\sigma(u,v)=\begin{pmatrix}\sqrt{v^2-1}\cos u\\\sqrt{v^2-1}\sin u\\v\end{pmatrix}$ とき，$D:(0,2\pi)\times\mathbb{R}$ とし，$\sigma(D)\subset S$ かつ $p=\sigma(\alpha,-2)$ であり $(\cos\alpha=\frac{1}{\sqrt{3}},\sin\alpha=\sqrt{\frac{2}{3}})$，$\sigma(D)$ は S でパラメータ表示された曲面片であるから，σ は S のパラメータ表示である

(3)

$$(\nabla f)(p)\cdot v=\begin{pmatrix}2\\2\sqrt{2}\\4\end{pmatrix}\cdot\begin{pmatrix}2\\-\sqrt{2}\\0\end{pmatrix}=0\text{ から，}\ v\in T_pS$$

$$\sigma_u=\begin{pmatrix}-\sqrt{v^2-1}\sin u\\\sqrt{v^2-1}\cos u\\0\end{pmatrix},\sigma_v=\begin{pmatrix}\frac{v}{\sqrt{v^2-1}}\cos u\\\frac{v}{\sqrt{v^2-1}}\sin u\\1\end{pmatrix}$$

$$\Longrightarrow\sigma_u(\alpha,-2)=\begin{pmatrix}-\sqrt{2}\\1\\0\end{pmatrix},\sigma_v(\alpha,-2)=\begin{pmatrix}-\frac{2}{3}\\-\frac{2}{3}\sqrt{2}\\1\end{pmatrix}$$

$$\Longrightarrow v=-\sqrt{2}\cdot\sigma_u(\alpha,-2)\text{ から，}\ \gamma(t)=\sigma\left(\alpha-\sqrt{2}t,-2\right)$$

1.8　P1.6

(1)

$f=x^2+y^2+z^2-4\sqrt{x^2+y^2}+3$ とすると，$f:C^\infty$ より，$S\cap\mathbb{R}^3=f^{-1}(\{0\})$

$(\nabla f)(p)=\begin{pmatrix}2x\left(1-\frac{2}{\sqrt{x^2+y^2}}\right)\\2y\left(1-\frac{2}{\sqrt{x^2+y^2}}\right)\\2z\end{pmatrix}=0$ となるのは $\begin{pmatrix}0\\\pm2\\0\end{pmatrix},\begin{pmatrix}\pm2\\0\\0\end{pmatrix}$ だけであるが，S に属していないから，$\forall p\in S,(\nabla f)(p)\neq0$

(2)

$x^2+y^2+z^2-4\sqrt{x^2+y^2}+3=0\iff\left(\sqrt{x^2+y^2}-2\right)^2+z^2=1$ から

$\sigma(u,v)=\begin{pmatrix}(2+\cos u)\cos v\\(2+\cos u)\sin v\\\sin u\end{pmatrix}$ とすれば，$D:=\mathbb{R}^2$ とし，$\sigma(D)\subset S$ かつ $p=\sigma\left(\frac{1}{2}\pi,\frac{7}{4}\pi\right)$ であり，$\sigma(D)$ は S でパラメータ表示された曲面片であるから，σ は S のパラメータ表示である

(3)

$$(\nabla f)(p)\cdot v=\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-5\\0\end{pmatrix}=0\Longrightarrow v\in T_pS$$

$$\sigma_u=\begin{pmatrix}-\sin u\cos v\\-\sin u\sin v\\\cos u\end{pmatrix},\sigma_v=\begin{pmatrix}-(2+\cos u)\sin v\\(2+\cos u)\cos v\\0\end{pmatrix}$$

$$\Longrightarrow\sigma_u\left(\frac{1}{2}\pi,\frac{7}{4}\pi\right)=\begin{pmatrix}-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\\0\end{pmatrix},\sigma_v\left(\frac{1}{2}\pi,\frac{7}{4}\pi\right)=\begin{pmatrix}\sqrt{2}\\\sqrt{2}\\0\end{pmatrix}$$

$$\Longrightarrow v=-3\sqrt{2}\cdot\sigma_u\left(\frac{1}{2}\pi,\frac{7}{4}\pi\right)-\sqrt{2}\cdot\sigma_v\left(\frac{1}{2}\pi,\frac{7}{4}\pi\right)\text{ から，}\ v=\sigma\left(\frac{1}{2}\pi-3\sqrt{2}t,\frac{7}{4}\pi-\sqrt{2}t\right)$$

1.9 P1.7

(1)

$f = x\sin\dfrac{z}{k} - y\cos\dfrac{z}{k}$ とすると，$f : C^\infty$ より，$S \cap \mathbb{R}^3 = f^{-1}(\{0\})$

$(\nabla f)(p) = \begin{pmatrix} \sin\dfrac{z}{k} \\ -\cos\dfrac{z}{k} \\ \dfrac{x}{k}\cos\dfrac{z}{k} + \dfrac{y}{k}\sin\dfrac{z}{k} \end{pmatrix}$ の第一第二成分が同時に 0 になれないから

$\forall p \in S, (\nabla f)(p) \neq 0$

以上より，S が正則曲面である

(2)

$\sin\dfrac{z}{k} = \cos\dfrac{z}{k} = 0$ となる z はないから

$\sigma(u, v) = \begin{pmatrix} u \\ u\tan\frac{v}{k} \\ v \end{pmatrix}$ （ここで $\cos\dfrac{z}{k}$ が 0 になると $y = u\cot\dfrac{z}{k}$ とすればいい）

$D := \mathbb{R}^2$ とし，$\sigma(D) \subset S$ かつ $p = \sigma\left(1, \dfrac{k\pi}{4}\right)$ であり，$\sigma(D)$ は S でパラメータ表示された曲面片であるから，σ は S のパラメータ表示である

(3)

$(\nabla f)(p) \cdot v = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{k} \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 2k \end{pmatrix} = 0$ から $v \in T_pS$

$\sigma_u = \begin{pmatrix} 1 \\ \tan\frac{v}{k} \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ \frac{2u}{k\left(1+\cos\frac{2v}{k}\right)} \\ 1 \end{pmatrix} \Longrightarrow \sigma_u\left(1, \dfrac{k\pi}{4}\right) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \sigma_v\left(1, \dfrac{k\pi}{4}\right) = \begin{pmatrix} 0 \\ \frac{2}{k} \\ 1 \end{pmatrix}$

$\Longrightarrow v = -3 \cdot \sigma_u\left(1, \dfrac{k\pi}{4}\right) + 2k \cdot \sigma_v\left(1, \dfrac{k\pi}{4}\right)$ から $\gamma(t) = \sigma\left(1 - 3t, \dfrac{k\pi}{4} + 2kt\right)$

1.10 P1.8

(1)

$f = 2z^2 - xy + yz - xz + 1$ とすると，$f : C^\infty$ より，$S \cap \mathbb{R}^3 = f^{-1}(\{0\})$

$(\nabla f)(p) = \begin{pmatrix} -y - z \\ -x + z \\ -x + y + 4z \end{pmatrix} = 0$ となるのは $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ だけであるが

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin S$ から，$\forall p \in S, (\nabla f)(p) \neq 0$

以上より，S が正則曲面である

(2)

$2z^2 - xy + yz - xz + 1 = 0$ より，y, z は同時に 0 になれない（そうしないと $1 = 0$）

よって，$\sigma(u,v) = \begin{pmatrix} \frac{2v^2+uv+1}{u+v} \\ u \\ v \end{pmatrix}$ とすれば，$D := \mathbb{R}^2$ とし，$\sigma(D) \subset S$ かつ $p = \sigma(-3,1)$ であり，$\sigma(D)$ は S でパラメータ表示された曲面片であるから，σ は S のパラメータ表示である

(3)

$(\nabla f)(p) \cdot v = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = 0$ から，$v \in T_pS$

$\sigma_v = \begin{pmatrix} -\frac{v^2+1}{(u+v)^2} \\ 1 \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} \frac{2v^2+4uv+u^2-1}{(u+v)^2} \\ 0 \\ 1 \end{pmatrix} \Longrightarrow \sigma_u(-3,1) = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \sigma_v(-3,1) = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$

$\Longrightarrow v = 2 \cdot \sigma_v(-3,1)$ から $\gamma(t) = \sigma(-3, 2t+1)$

2 §2

2.1 E2.1

(1)

$$\sigma_u = \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} -u\sin v \\ u\cos v \\ k \end{pmatrix}, \begin{cases} E = \sigma_u \cdot \sigma_u = 1 \\ F = \sigma_u \cdot \sigma_v = 0 \\ G = \sigma_v \cdot \sigma_v = u^2 + k^2 \end{cases}$$

(2)

$$f = x\sin\frac{z}{k} - y\cos\frac{z}{k}, \nabla f = \begin{pmatrix} \sin\frac{z}{k} \\ -\cos\frac{z}{k} \\ \frac{1}{k}\left(x\cos\frac{z}{k} + y\sin\frac{z}{k}\right) \end{pmatrix}, \|\nabla f\|^2 = 1 + \frac{1}{k^2}\left(x\cos\frac{z}{k} + y\sin\frac{z}{k}\right)^2$$

$$\frac{\nabla f}{\|\nabla f\|} = \frac{1}{\sqrt{k^2 + \left(x\cos\frac{z}{k} + y\sin\frac{z}{k}\right)^2}} \begin{pmatrix} k\sin\frac{z}{k} \\ -k\cos\frac{z}{k} \\ x\cos\frac{z}{k} + y\sin\frac{z}{k} \end{pmatrix}$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix} \times \begin{pmatrix} -u\sin v \\ u\cos v \\ k \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} k\sin v \\ -k\cos v \\ u \end{pmatrix} \tag{2}$$

$$\|\sigma_u \times \sigma_v\| = \sqrt{k^2 + u^2} \tag{3}$$

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{k^2+u^2}} \begin{pmatrix} k\sin v \\ -k\cos v \\ u \end{pmatrix} = \frac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|} \tag{4}$$

(3)

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} -\sin v \\ \cos v \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} -u\cos v \\ -u\sin v \\ 0 \end{pmatrix}$$

$$\begin{cases} L = \sigma_{uu} \cdot \mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} k\sin v \\ -k\cos v \\ u \end{pmatrix} \frac{1}{\sqrt{k^2+u^2}} = 0 \\ M = \sigma_{uv} \times \mathbf{n} = \begin{pmatrix} -\sin v \\ \cos v \\ 0 \end{pmatrix} \cdot \begin{pmatrix} k\sin v \\ -k\cos v \\ u \end{pmatrix} \frac{1}{\sqrt{k^2+u^2}} = -\frac{k}{\sqrt{k^2+u^2}} \\ N = \sigma_{vv} \cdot \mathbf{n} = \begin{pmatrix} -u\cos v \\ -u\sin v \\ 0 \end{pmatrix} \cdot \begin{pmatrix} k\sin v \\ -k\cos v \\ u \end{pmatrix} \frac{1}{\sqrt{k^2+u^2}} = 0 \end{cases}$$

2.2 E2.2

(1)

$$\sigma_u = \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix}, \sigma_v = \begin{pmatrix} -(R+r\cos u)\sin v \\ (R+r\cos u)\cos v \\ 0 \end{pmatrix}, \begin{cases} E=\sigma_u\cdot\sigma_u=r^2 \\ F=\sigma_u\cdot\sigma_v=0 \\ G=\sigma_v\cdot\sigma_v=(R+r\cos u)^2 \end{cases}$$

(2)

$$f=\left(\sqrt{x^2+y^2}-R\right)^2+z^2-r^2, \frac{\nabla f}{\|\nabla f\|}=\frac{1}{\sqrt{\left(\sqrt{x^2+y^2}-R\right)^2+z^2}}\begin{pmatrix} \frac{x\left(\sqrt{x^2+y^2}-R\right)}{\sqrt{x^2+y^2}} \\ \frac{y\left(\sqrt{x^2+y^2}-R\right)}{\sqrt{x^2+y^2}} \\ z \end{pmatrix}$$

$$\sigma_u\times\sigma_v=\begin{pmatrix} -r(R+r\cos u)\cos u\cos v \\ -r(R+r\cos u)\cos u\sin v \\ -r(R+r\cos u)\sin u \end{pmatrix} \tag{5}$$

$$\|\sigma_u\times\sigma_v\|=r(R+r\cos u) \tag{6}$$

$$\mathbf{n}=\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}=\begin{pmatrix} -\cos u\cos v \\ -\cos u\sin v \\ -\sin u \end{pmatrix}=-\frac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|} \tag{7}$$

から，$\mathbf{n}=-\frac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|}$ は正の向きにする単位法ベクトル場

(3)

$$\sigma_{uu}=\begin{pmatrix} -r\cos u\cos v \\ -r\cos u\sin v \\ -r\sin u \end{pmatrix}, \sigma_{uv}=\begin{pmatrix} r\sin u\sin v \\ -r\sin u\cos v \\ 0 \end{pmatrix}, \sigma_{vv}=\begin{pmatrix} -(R+r\cos u)\cos v \\ -(R+r\cos u)\sin v \\ 0 \end{pmatrix}$$

$$\begin{cases} L=\sigma_{uu}\cdot\mathbf{n}=\begin{pmatrix} -r\cos u\cos v \\ -r\cos u\sin v \\ -r\sin u \end{pmatrix}\cdot\begin{pmatrix} -\cos u\cos v \\ -\cos u\sin v \\ -\sin u \end{pmatrix}=r \\ M=\sigma_{uv}\cdot\mathbf{n}=\begin{pmatrix} r\sin u\sin v \\ -r\sin u\cos v \\ 0 \end{pmatrix}\cdot\begin{pmatrix} -\cos u\cos v \\ -\cos u\sin v \\ -\sin u \end{pmatrix}=0 \\ N=\sigma_{vv}\cdot\mathbf{n}=\begin{pmatrix} -(R+r\cos u)\cos v \\ -(R+r\cos u)\sin v \\ 0 \end{pmatrix}\cdot\begin{pmatrix} -\cos u\cos v \\ -\cos u\sin v \\ -\sin u \end{pmatrix}=(R+r\cos u)\cos u \end{cases}$$

2.3　P2.1

(1)

$$\sigma_u = \begin{pmatrix} a\cos u\cos v \\ b\cos u\sin v \\ -c\sin u \end{pmatrix}, \sigma_v = \begin{pmatrix} -a\sin u\sin v \\ b\sin u\cos v \\ 0 \end{pmatrix}$$

$$\begin{cases} E = \sigma_u \cdot \sigma_u = a^2\cos^2 u\cos^2 v + b^2\cos^2 u\sin^2 v + c\sin^2 u \\ F = \sigma_u \cdot \sigma_v = \left(b^2 - a^2\right)\sin u\cos u\sin v\cos v \\ G = \sigma_v \cdot \sigma_v = a^2\sin^2 u\sin^2 v + b^2\sin^2 u\cos^2 v \end{cases}$$

(2)

$$f = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1, \nabla f = \begin{pmatrix} \frac{2x}{a^2} \\ \frac{2y}{b^2} \\ \frac{2z}{c^2} \end{pmatrix}, \|\nabla f\| = 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

$$\frac{\nabla f}{\|\nabla f\|} = \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \\ \frac{z}{c^2} \end{pmatrix}$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} bc\sin^2 u\cos v \\ ac\sin^2 u\sin v \\ ab\sin u\cos u \end{pmatrix}$$

$$\|\sigma_u \times \sigma_v\| = \sqrt{b^2c^2\sin^4 u\cos^2 v + a^2c^2\sin^4 u\sin^2 v + a^2b^2\sin^2 u\cos^2 u}$$

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{b^2c^2\sin^4 u\cos^2 v + a^2c^2\sin^4 u\sin^2 v + a^2b^2\sin^2 u\cos^2 u}}\begin{pmatrix} bc\sin^2 u\cos v \\ ac\sin^2 u\sin v \\ ab\sin u\cos u \end{pmatrix}$$

$= \dfrac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|}$ は σ が正の向きになるものである

(3)

$$\sigma_{uu} = \begin{pmatrix} -a\sin u\cos v \\ -b\sin u\sin v \\ -c\cos u \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} -a\cos u\sin v \\ b\cos u\cos v \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} -a\sin u\cos v \\ -b\sin u\sin v \\ 0 \end{pmatrix}$$

$$L = \sigma_{uu} \cdot \mathbf{n} \tag{8}$$

$$= \frac{1}{\sqrt{b^2c^2\sin^4 u\cos^2 v + a^2c^2\sin^4 u\sin^2 v + a^2b^2\sin^2 u\cos^2 u}}\begin{pmatrix} -a\sin u\cos v \\ -b\sin u\sin v \\ -c\cos u \end{pmatrix} \cdot \begin{pmatrix} bc\sin^2 u\cos v \\ ac\sin^2 u\sin v \\ ab\sin u\cos u \end{pmatrix} \tag{9}$$

$$= -\frac{abc\sin u}{\sqrt{b^2c^2\sin^4 u\cos^2 v + a^2c^2\sin^4 u\sin^2 v + a^2b^2\sin^2 u\cos^2 u}}\left(\sin^2 u + \cos^2 u\right) \tag{10}$$

$$= -\frac{abc\sin u}{\sqrt{b^2c^2\sin^4 u\cos^2 v + a^2c^2\sin^4 u\sin^2 v + a^2b^2\sin^2 u\cos^2 u}} \tag{11}$$

$$M = \sigma_{uv} \cdot \mathbf{n} \tag{12}$$

$$= \frac{1}{\sqrt{b^2c^2\sin^4 u\cos^2 v + a^2c^2\sin^4 u\sin^2 v + a^2b^2\sin^2 u\cos^2 u}} \begin{pmatrix} -a\cos u\sin v \\ b\cos u\cos v \\ 0 \end{pmatrix} \cdot \begin{pmatrix} bc\sin^2 u\cos v \\ ac\sin^2 u\sin v \\ ab\sin u\cos u \end{pmatrix} \tag{13}$$

$$= 0 \tag{14}$$

$$N = \sigma_{vv} \cdot \mathbf{n} \tag{15}$$

$$= \frac{1}{\sqrt{b^2c^2\sin^4 u\cos^2 v + a^2c^2\sin^4 u\sin^2 v + a^2b^2\sin^2 u\cos^2 u}} \begin{pmatrix} -a\sin u\cos v \\ -b\sin u\sin v \\ 0 \end{pmatrix} \cdot \begin{pmatrix} bc\sin^2 u\cos v \\ ac\sin^2 u\sin v \\ ab\sin u\cos u \end{pmatrix} \tag{16}$$

$$= -\frac{abc\sin^3 u}{\sqrt{b^2c^2\sin^4 u\cos^2 v + a^2c^2\sin^4 u\sin^2 v + a^2b^2\sin^2 u\cos^2 u}} \tag{17}$$

2.4 P2.2

(1)

$$\sigma_u = \begin{pmatrix} a\sinh u\cos v \\ b\sinh u\sin v \\ c\cosh u \end{pmatrix}, \sigma_v = \begin{pmatrix} -a\cosh u\sin v \\ b\cosh u\cos v \\ 0 \end{pmatrix}$$

$$\begin{cases} E = \sigma_u \cdot \sigma_u = a^2\sinh^2 u\cos^2 v + b^2\sinh^2 u\sin^2 v + c\cosh^2 u \\ F = \sigma_u \cdot \sigma_v = \left(b^2 - a^2\right)\sinh u\cosh u\sin v\cos v \\ G = \sigma_v \cdot \sigma_v = a^2\cosh^2 u\sin^2 v + b^2\cosh^2 u\cos^2 v \end{cases}$$

(2)

$$f = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1, \nabla f = \begin{pmatrix} \frac{2x}{a^2} \\ \frac{2y}{b^2} \\ -\frac{2z}{c^2} \end{pmatrix}, \|\nabla f\| = 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} -bc\cosh^2 u\cos v \\ -ac\cosh^2 u\sin v \\ ab\sinh u\cosh u \end{pmatrix}$$

$$\|\sigma_u \times \sigma_v\| = \sqrt{b^2c^2\cosh^4 u\cos^2 v + a^2c^2\cosh^4 u\sin^2 v + a^2b^2\sinh^2 u\cosh^2 u}$$

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{b^2c^2\cosh^4 u\cos^2 v + a^2c^2\cosh^4 u\sin^2 v + a^2b^2\sinh^2 u\cosh^2 u}} \begin{pmatrix} -bc\cosh^2 u\cos v \\ -ac\cosh^2 u\sin v \\ ab\sinh u\cosh u \end{pmatrix}$$

$= -\frac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|}$ は σ が正の向きになる単位法ベクトル場

(3)

$$\sigma_{uu} = \begin{pmatrix} a\cosh u\cos v \\ b\cosh u\sin v \\ c\sinh u \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} -a\sinh u\sin v \\ b\sinh u\cos v \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} -a\cosh u\cos v \\ -b\cosh u\sin v \\ 0 \end{pmatrix}$$

$$\begin{aligned}
L &= \sigma_{uu}\cdot\mathbf{n} && (18)\\
&= \begin{pmatrix} a\cosh u\cos v \\ b\cosh u\sin v \\ c\sinh u \end{pmatrix}\cdot\begin{pmatrix} -bc\cosh^2 u\cos v \\ -ac\cosh^2 u\sin v \\ ab\sinh u\cosh u \end{pmatrix} && (19)\\
&\cdot\frac{1}{\sqrt{b^2c^2\cosh^4 u\cos^2 v + a^2c^2\cosh^4 u\sin^2 v + a^2b^2\sinh^2 u\cosh^2 u}} && (20)\\
&= -\frac{abc\cosh u}{\sqrt{b^2c^2\cosh^4 u\cos^2 v + a^2c^2\cosh^4 u\sin^2 v + a^2b^2\sinh^2 u\cosh^2 u}} && (21)\\
M &= \sigma_{uv}\cdot\mathbf{n} && (22)\\
&= \frac{1}{\sqrt{b^2c^2\cosh^4 u\cos^2 v + a^2c^2\cosh^4 u\sin^2 v + a^2b^2\sinh^2 u\cosh^2 u}} && (23)\\
&\cdot\begin{pmatrix} -a\sinh u\sin v \\ b\sinh u\cos v \\ 0 \end{pmatrix}\cdot\begin{pmatrix} -bc\cosh^2 u\cos v \\ -ac\cosh^2 u\sin v \\ ab\sinh u\cosh u \end{pmatrix} && (24)\\
&= 0 && (25)\\
N &= \sigma_{vv}\cdot\mathbf{n} && (26)\\
&= \frac{1}{\sqrt{b^2c^2\cosh^4 u\cos^2 v + a^2c^2\cosh^4 u\sin^2 v + a^2b^2\sinh^2 u\cosh^2 u}} && (27)\\
&\cdot\begin{pmatrix} -a\cosh u\cos v \\ -b\cosh u\sin v \\ 0 \end{pmatrix}\cdot\begin{pmatrix} -bc\cosh^2 u\cos v \\ -ac\cosh^2 u\sin v \\ ab\sinh u\cosh u \end{pmatrix} && (28)\\
&= \frac{abc\cosh^3 u}{\sqrt{b^2c^2\cosh^4 u\cos^2 v + a^2c^2\cosh^4 u\sin^2 v + a^2b^2\sinh^2 u\cosh^2 u}} && (29)
\end{aligned}$$

2.5 P2.3

(1)

$$\sigma_+ : \sigma_u = \begin{pmatrix} a \\ 0 \\ \frac{cu}{\sqrt{u^2+v^2+1}} \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ b \\ \frac{cv}{\sqrt{u^2+v^2+1}} \end{pmatrix}, \sigma_- : \sigma_u = \begin{pmatrix} a \\ 0 \\ -\frac{cu}{\sqrt{u^2+v^2+1}} \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ b \\ -\frac{cv}{\sqrt{u^2+v^2+1}} \end{pmatrix}$$

$$\begin{cases} E_+ = \sigma_u\cdot\sigma_u = a^2 + \dfrac{c^2u^2}{u^2+v^2+1} \\ F_+ = \sigma_u\cdot\sigma_v = \dfrac{c^2uv}{u^2+v^2+1} \\ G_+ = \sigma_v\cdot\sigma_v = b^2 + \dfrac{c^2v^2}{u^2+v^2+1} \end{cases}, \begin{cases} E_- = \sigma_u\cdot\sigma_u = a^2 + \dfrac{c^2u^2}{u^2+v^2+1} \\ F_- = \sigma_u\cdot\sigma_v = \dfrac{c^2uv}{u^2+v^2+1} \\ G_- = \sigma_v\cdot\sigma_v = b^2 + \dfrac{c^2v^2}{u^2+v^2+1} \end{cases}$$

(2)

σ_+ :

$$\sigma_u \times \sigma_v = \begin{pmatrix} -\frac{bcu}{\sqrt{u^2+v^2+1}} \\ -\frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}$$

σ_- :

$$\sigma_u \times \sigma_v = \begin{pmatrix} \frac{bcu}{\sqrt{u^2+v^2+1}} \\ \frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}$$

$$\nabla f = \begin{pmatrix} \frac{2x}{a^2} \\ \frac{2y}{b^2} \\ -\frac{2z}{c^2} \end{pmatrix}, \|\nabla f\| = 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

よって，$\mathbf{n}_+ = \frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}} \begin{pmatrix} -\frac{bcu}{\sqrt{u^2+v^2+1}} \\ -\frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix} = -\frac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|}$

$\mathbf{n}_- = \frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}} \begin{pmatrix} \frac{bcu}{\sqrt{u^2+v^2+1}} \\ \frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix} = \frac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|}$ はそれぞれ σ_+, σ_- が正の向きになる単位法ベクトル場である

(3)

σ_+ :

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ \frac{c\left(v^2+1\right)}{(u^2+v^2+1)^{\frac{3}{2}}} \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} 0 \\ 0 \\ -\frac{cuv}{(u^2+v^2+1)^{\frac{3}{2}}} \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} 0 \\ 0 \\ \frac{c\left(u^2+1\right)}{(u^2+v^2+1)^{\frac{3}{2}}} \end{pmatrix}$$

$$L = \sigma_{uu} \cdot \mathbf{n} \tag{30}$$

$$= \frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}} \begin{pmatrix} 0 \\ 0 \\ \frac{c\left(v^2+1\right)}{(u^2+v^2+1)^{\frac{3}{2}}} \end{pmatrix} \cdot \begin{pmatrix} -\frac{bcu}{\sqrt{u^2+v^2+1}} \\ -\frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix} \tag{31}$$

$$= \frac{abc\left(v^2+1\right)}{(u^2+v^2+1)\sqrt{b^2c^2u^2+a^2c^2v^2+a^2b^2\left(u^2+v^2+1\right)}} \tag{32}$$

$$M = \sigma_{uv} \cdot \mathbf{n} \tag{33}$$

$$= \frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}} \begin{pmatrix} 0 \\ 0 \\ -\frac{cuv}{(u^2+v^2+1)^{\frac{3}{2}}} \end{pmatrix} \cdot \begin{pmatrix} -\frac{bcu}{\sqrt{u^2+v^2+1}} \\ -\frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix} \tag{34}$$

$$= -\frac{abcuv}{(u^2+v^2+1)\sqrt{b^2c^2u^2+a^2c^2v^2+a^2b^2\left(u^2+v^2+1\right)}} \tag{35}$$

$$N = \sigma_{vv} \cdot \mathbf{n} \tag{36}$$

$$= \frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}} \begin{pmatrix} 0 \\ 0 \\ \frac{c\left(u^2+1\right)}{(u^2+v^2+1)^{\frac{3}{2}}} \end{pmatrix} \cdot \begin{pmatrix} -\frac{bcu}{\sqrt{u^2+v^2+1}} \\ -\frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix} \tag{37}$$

$$= \frac{abc\left(u^2+1\right)}{(u^2+v^2+1)\sqrt{b^2c^2u^2+a^2c^2v^2+a^2b^2\left(u^2+v^2+1\right)}} \tag{38}$$

σ_- :

$$\sigma_{uu}=\begin{pmatrix}0\\0\\-\frac{c\left(v^2+1\right)}{(u^2+v^2+1)^{\frac{3}{2}}}\end{pmatrix},\sigma_{uv}=\begin{pmatrix}0\\0\\\frac{cuv}{(u^2+v^2+1)^{\frac{3}{2}}}\end{pmatrix},\sigma_{vv}=\begin{pmatrix}0\\0\\-\frac{c\left(u^2+1\right)}{(u^2+v^2+1)^{\frac{3}{2}}}\end{pmatrix}$$

$$L=\sigma_{uu}\cdot\mathbf{n} \tag{39}$$

$$=\frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1}+\frac{a^2c^2v^2}{u^2+v^2+1}+a^2b^2}}\begin{pmatrix}0\\0\\-\frac{c\left(v^2+1\right)}{(u^2+v^2+1)^{\frac{3}{2}}}\end{pmatrix}\cdot\begin{pmatrix}\frac{bcu}{\sqrt{u^2+v^2+1}}\\\frac{acv}{\sqrt{u^2+v^2+1}}\\ab\end{pmatrix} \tag{40}$$

$$=-\frac{abc\left(v^2+1\right)}{(u^2+v^2+1)\sqrt{b^2c^2u^2+a^2c^2v^2+a^2b^2\left(u^2+v^2+1\right)}} \tag{41}$$

$$M=\sigma_{uv}\cdot\mathbf{n} \tag{42}$$

$$=\frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1}+\frac{a^2c^2v^2}{u^2+v^2+1}+a^2b^2}}\begin{pmatrix}0\\0\\\frac{cuv}{(u^2+v^2+1)^{\frac{3}{2}}}\end{pmatrix}\cdot\begin{pmatrix}\frac{bcu}{\sqrt{u^2+v^2+1}}\\\frac{acv}{\sqrt{u^2+v^2+1}}\\ab\end{pmatrix} \tag{43}$$

$$=\frac{abcuv}{(u^2+v^2+1)\sqrt{b^2c^2u^2+a^2c^2v^2+a^2b^2\left(u^2+v^2+1\right)}} \tag{44}$$

$$N=\sigma_{vv}\cdot\mathbf{n} \tag{45}$$

$$=\frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1}+\frac{a^2c^2v^2}{u^2+v^2+1}+a^2b^2}}\begin{pmatrix}0\\0\\-\frac{c\left(u^2+1\right)}{(u^2+v^2+1)^{\frac{3}{2}}}\end{pmatrix}\cdot\begin{pmatrix}\frac{bcu}{\sqrt{u^2+v^2+1}}\\\frac{acv}{\sqrt{u^2+v^2+1}}\\ab\end{pmatrix} \tag{46}$$

$$=-\frac{abc\left(u^2+1\right)}{(u^2+v^2+1)\sqrt{b^2c^2u^2+a^2c^2v^2+a^2b^2\left(u^2+v^2+1\right)}} \tag{47}$$

2.6 P2.4

(1)

$$\sigma_u=\begin{pmatrix}1\\0\\\frac{2u}{a^2}\end{pmatrix},\sigma_v=\begin{pmatrix}0\\1\\\frac{2v}{b^2}\end{pmatrix},\begin{cases}E=\sigma_u\cdot\sigma_u=1+\dfrac{4u^2}{a^4}\\F=\sigma_u\cdot\sigma_v=\dfrac{4uv}{a^2b^2}\\G=\sigma_v\cdot\sigma_v=1+\dfrac{4v^2}{b^4}\end{cases}$$

(2)

$$f=\frac{x^2}{a^2}+\frac{y^2}{b^2}-z,\nabla f=\begin{pmatrix}\frac{2x}{a^2}\\\frac{2y}{b^2}\\-1\end{pmatrix}$$

$$\sigma_u\times\sigma_v=\begin{pmatrix}-\frac{2u}{a^2}\\-\frac{2v}{b^2}\\1\end{pmatrix},\|\sigma_u\times\sigma_v\|=\sqrt{\frac{4u^2}{a^4}+\frac{4v^2}{b^4}+1}$$

$$\mathbf{n}=\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}=\frac{1}{\sqrt{\frac{4u^2}{a^4}+\frac{4v^2}{b^4}+1}}\begin{pmatrix}-\frac{2u}{a^2}\\-\frac{2v}{b^2}\\1\end{pmatrix}=-\frac{(\nabla f)\left(\sigma\left(u,v\right)\right)}{\|(\nabla f)\left(\sigma\left(u,v\right)\right)\|}$$

(3)

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{a^2} \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{b^2} \end{pmatrix}$$

$$L = \sigma_{uu} \cdot \mathbf{n} \tag{48}$$

$$= \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} 0 \\ 0 \\ \frac{2}{a^2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{2u}{a^2} \\ -\frac{2v}{b^2} \\ 1 \end{pmatrix} \tag{49}$$

$$= \frac{2}{a^2\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \tag{50}$$

$$M = \sigma_{uv} \cdot \mathbf{n} \tag{51}$$

$$= \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\frac{2u}{a^2} \\ -\frac{2v}{b^2} \\ 1 \end{pmatrix} \tag{52}$$

$$= 0 \tag{53}$$

$$N = \sigma_{vv} \cdot \mathbf{n} \tag{54}$$

$$= \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} 0 \\ 0 \\ \frac{2}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{2u}{a^2} \\ -\frac{2v}{b^2} \\ 1 \end{pmatrix} \tag{55}$$

$$= \frac{2}{b^2\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \tag{56}$$

2.7 P2.5

(1)

$$\sigma_u = \begin{pmatrix} 1 \\ 0 \\ \frac{2u}{a^2} \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 1 \\ -\frac{2v}{b^2} \end{pmatrix}, \begin{cases} E = \sigma_u \cdot \sigma_u = 1 + \dfrac{4u^2}{a^4} \\ F = \sigma_u \cdot \sigma_v = -\dfrac{4uv}{a^2b^2} \\ G = \sigma_v \cdot \sigma_v = 1 + \dfrac{4v^2}{b^4} \end{cases}$$

(2)

$$f = \frac{x^2}{a^2} - \frac{y^2}{b^2} - z, \nabla f = \begin{pmatrix} \frac{2x}{a^2} \\ -\frac{2y}{b^2} \\ -1 \end{pmatrix}, \|\nabla f\| = \sqrt{\frac{4x^2}{a^4} + \frac{4y^2}{b^4} + 1}$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} -\frac{2u}{a^2} \\ \frac{2v}{b^2} \\ 1 \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}$$

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} -\frac{2u}{a^2} \\ \frac{2v}{b^2} \\ 1 \end{pmatrix} = -\frac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|}$$

(3)

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{a^2} \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} 0 \\ 0 \\ -\frac{2}{b^2} \end{pmatrix}$$

$$L = \sigma_{uu} \cdot \mathbf{n} \tag{57}$$

$$= \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} 0 \\ 0 \\ \frac{2}{a^2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{2u}{a^2} \\ \frac{2v}{b^2} \\ 1 \end{pmatrix} \tag{58}$$

$$= \frac{2}{a^2\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \tag{59}$$

$$M = \sigma_{uv} \cdot \mathbf{n} \tag{60}$$

$$= \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\frac{2u}{a^2} \\ \frac{2v}{b^2} \\ 1 \end{pmatrix} \tag{61}$$

$$= 0 \tag{62}$$

$$N = \sigma_{vv} \cdot \mathbf{n} \tag{63}$$

$$= \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} 0 \\ 0 \\ -\frac{2}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{2u}{a^2} \\ \frac{2v}{b^2} \\ 1 \end{pmatrix} \tag{64}$$

$$= -\frac{2}{b^2\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \tag{65}$$

2.8 P2.6

(1)

$$\sigma_u = \begin{pmatrix} 1 \\ 3u^2 \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{cases} E = \sigma_u \cdot \sigma_u = 9u^4 + 1 \\ F = \sigma_u \cdot \sigma_v = 0 \\ G = \sigma_v \cdot \sigma_v = 1 \end{cases}$$

(2)

$$f = x^3 - y, \nabla f = \begin{pmatrix} 3x^2 \\ -1 \\ 0 \end{pmatrix}, \|\nabla f\| = \sqrt{9x^4 + 1}$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} 3u^2 \\ -1 \\ 0 \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \sqrt{9u^4 + 1}$$

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{9u^4 + 1}} \begin{pmatrix} 3u^2 \\ -1 \\ 0 \end{pmatrix} = \frac{(\nabla f)(\sigma(u, v))}{\|(\nabla f)(\sigma(u, v))\|}$$

(3)

$$\sigma_{uu}=\begin{pmatrix}0\\6u\\0\end{pmatrix},\sigma_{uv}=\begin{pmatrix}0\\0\\0\end{pmatrix},\sigma_{vv}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{aligned}
L&=\sigma_{uu}\cdot\mathbf{n} &(66)\\
&=\frac{1}{\sqrt{9u^4+1}}\begin{pmatrix}0\\6u\\0\end{pmatrix}\cdot\begin{pmatrix}3u^2\\-1\\0\end{pmatrix} &(67)\\
&=-\frac{6u}{\sqrt{9u^4+1}} &(68)\\
M&=\sigma_{uv}\cdot\mathbf{n} &(69)\\
&=\frac{1}{\sqrt{9u^4+1}}\begin{pmatrix}0\\0\\0\end{pmatrix}\cdot\begin{pmatrix}3u^2\\-1\\0\end{pmatrix} &(70)\\
&=0 &(71)\\
N&=\sigma_{vv}\cdot\mathbf{n} &(72)\\
&=\frac{1}{\sqrt{9u^4+1}}\begin{pmatrix}0\\0\\0\end{pmatrix}\cdot\begin{pmatrix}3u^2\\-1\\0\end{pmatrix} &(73)\\
&=0 &(74)
\end{aligned}$$

2.9 P2.7

(1)

$$\sigma_u=\begin{pmatrix}1\\0\\\tan u\end{pmatrix},\sigma_v=\begin{pmatrix}0\\1\\-\tan v\end{pmatrix},\begin{cases}E=\sigma_u\cdot\sigma_u=\dfrac{1}{\cos^2 u}\\F=\sigma_u\cdot\sigma_v=-\tan u\tan v\\G=\sigma_v\cdot\sigma_v=\dfrac{1}{\cos^2 v}\end{cases}$$

(2)

$$f=e^z\cos x-\cos y,\nabla f=\begin{pmatrix}-e^z\sin x\\\sin y\\e^z\cos x\end{pmatrix},\|\nabla f\|=\sqrt{e^{2z}+\sin^2 y}$$

$$\sigma_u\times\sigma_v=\begin{pmatrix}-\tan u\\\tan v\\1\end{pmatrix},\|\sigma_u\times\sigma_v\|=\sqrt{\tan^2 u+\tan^2 v+1}$$

$$\mathbf{n}=\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}=\frac{1}{\sqrt{\tan^2 u+\tan^2 v+1}}\begin{pmatrix}-\tan u\\\tan v\\1\end{pmatrix}=\frac{(\nabla f)\left(\sigma\left(u,v\right)\right)}{\|(\nabla f)\left(\sigma\left(u,v\right)\right)\|}$$

(3)

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\cos^2 u} \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\cos^2 v} \end{pmatrix}$$

$$L = \sigma_{uu} \cdot \mathbf{n} \tag{75}$$

$$= \frac{1}{\sqrt{\tan^2 u + \tan^2 v + 1}} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\cos^2 u} \end{pmatrix} \cdot \begin{pmatrix} -\tan u \\ \tan v \\ 1 \end{pmatrix} \tag{76}$$

$$= \frac{1}{\cos^2 u \sqrt{\tan^2 u + \tan^2 v + 1}} \tag{77}$$

$$M = \sigma_{uv} \cdot \mathbf{n} \tag{78}$$

$$= \frac{1}{\sqrt{\tan^2 u + \tan^2 v + 1}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\tan u \\ \tan v \\ 1 \end{pmatrix} \tag{79}$$

$$= 0 \tag{80}$$

$$N = \sigma_{vv} \cdot \mathbf{n} \tag{81}$$

$$= \frac{1}{\sqrt{\tan^2 u + \tan^2 v + 1}} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\cos^2 v} \end{pmatrix} \cdot \begin{pmatrix} -\tan u \\ \tan v \\ 1 \end{pmatrix} \tag{82}$$

$$= -\frac{1}{\cos^2 v \sqrt{\tan^2 u + \tan^2 v + 1}} \tag{83}$$

3 §3

3.1 E3.1

$\gamma(0) = \mathbf{p}, \dfrac{\mathrm{d}\gamma}{\mathrm{d}t}(0) = \mathbf{v}$ となる $\gamma(t) \subset S$ を取り，$\mathbf{n}(\mathbf{p}) = \dfrac{1}{r}\mathbf{p}$ より，$\mathbf{n}(\gamma(t)) = \dfrac{1}{r}\gamma(t)$ で，$\left.\dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{n}(\gamma(t))\right|_{t=0} = \dfrac{1}{r}\dfrac{\mathrm{d}\gamma}{\mathrm{d}t}(0) = \dfrac{1}{r}\mathbf{v}$ から，$\sum_{\mathbf{p}}(\mathbf{v}) = -\left.\dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{n}(\gamma(t))\right|_{t=0} = -\dfrac{1}{r}\mathbf{v}$

3.2 E3.2

$$\sigma_u = \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} -u\sin v \\ u\cos v \\ k \end{pmatrix}, \begin{cases} E = \sigma_u \cdot \sigma_u = 1 \\ F = \sigma_u \cdot \sigma_v = 0 \\ G = \sigma_v \cdot \sigma_v = u^2 + k^2 \end{cases}$$

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{u^2+k^2}} \begin{pmatrix} k\sin v \\ -k\cos v \\ u \end{pmatrix}$$

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} -\sin v \\ \cos v \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} -u\cos v \\ -u\sin v \\ 0 \end{pmatrix}$$

$$L = \sigma_{uu} \cdot \mathbf{n} = 0 \tag{84}$$

$$M = \sigma_{uv} \cdot \mathbf{n} = -\frac{k}{\sqrt{u^2+k^2}} \tag{85}$$

$$N = \sigma_{vv} \cdot \mathbf{n} = 0 \tag{86}$$

以上より

$$K = \frac{LN - M^2}{EG - F^2} \tag{87}$$

$$= -\frac{\frac{k^2}{u^2+k^2}}{u^2+k^2} \tag{88}$$

$$= -\frac{k^2}{(u^2+k^2)^2} \tag{89}$$

$$H = \frac{GL - 2FM + EN}{2(EG - F^2)} \tag{90}$$

$$= \frac{0}{2(u^2+k^2)} = 0 \tag{91}$$

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K} \tag{92}$$

$$= \pm \frac{k}{u^2+k^2} \tag{93}$$

3.3 E3.3

$f(x,y,z) := \left(\sqrt{x^2+y^2} - R\right)^2 + z^2 - r^2$ で $\nabla f = \begin{pmatrix} 2\left(\sqrt{x^2+y^2} - R\right)\frac{x}{\sqrt{x^2+y^2}} \\ 2\left(\sqrt{x^2+y^2} - R\right)\frac{y}{\sqrt{x^2+y^2}} \\ 2z \end{pmatrix}$

$T_{\mathbf{p}}S = \left\{\mathbf{v} \in \mathbb{R}^3 \middle| (\nabla f)(\mathbf{p}) \cdot \mathbf{v} = 0\right\}$ から

$$T_{\mathbf{p}}S := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| 2\left(\sqrt{x^2+y^2}-R\right)\sqrt{x^2+y^2}+2z^2=0 \right\} = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in (0,2\pi)^2 \middle| 2Rr\cos u + 2r^2 = 0 \right\}$$

$\sigma_u = \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix}, \sigma_v = \begin{pmatrix} -(R+r\cos u)\sin v \\ (R+r\cos u)\cos v \\ 0 \end{pmatrix}$ から

$$\sigma_u \times \sigma_v = \begin{pmatrix} -r(R+r\cos u)\cos u\cos v \\ -r(R+r\cos u)\cos u\sin v \\ -r(R+r\cos u)\sin u \end{pmatrix} \tag{94}$$

$$\|\sigma_u \times \sigma_v\| = r(R+r\cos u) \tag{95}$$

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \begin{pmatrix} -\cos u\cos v \\ -\cos u\sin v \\ -\sin u \end{pmatrix} = -\frac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|} \tag{96}$$

$$= -\frac{1}{\sqrt{\left(\sqrt{x^2+y^2}-R\right)^2+z^2}} \begin{pmatrix} \left(\sqrt{x^2+y^2}-R\right)\frac{x}{\sqrt{x^2+y^2}} \\ \left(\sqrt{x^2+y^2}-R\right)\frac{y}{\sqrt{x^2+y^2}} \\ z \end{pmatrix} \tag{97}$$

$$= -\frac{1}{r} \begin{pmatrix} \left(\sqrt{x^2+y^2}-R\right)\frac{x}{\sqrt{x^2+y^2}} \\ \left(\sqrt{x^2+y^2}-R\right)\frac{y}{\sqrt{x^2+y^2}} \\ z \end{pmatrix} \tag{98}$$

$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in T_{\mathbf{p}}T_{R,r}$ とし，$\gamma(0)=\mathbf{p}, \frac{\mathrm{d}\gamma}{\mathrm{d}t}(0)=\mathbf{v}$ をみたす C^∞ 曲線 $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$ を考えよう．
γ を法ベクトル $\mathbf{n}$ に代入すると

$$\mathbf{n}(\gamma(t)) = -\frac{1}{r} \begin{pmatrix} \left(\sqrt{\gamma_1^2+\gamma_2^2}-R\right)\frac{\gamma_1}{\sqrt{\gamma_1^2+\gamma_2^2}} \\ \left(\sqrt{\gamma_1^2+\gamma_2^2}-R\right)\frac{\gamma_2}{\sqrt{\gamma_1^2+\gamma_2^2}} \\ \gamma_3 \end{pmatrix} \tag{99}$$

ただし，ここで $\gamma_1, \gamma_2, \gamma_3$ はすべて $\gamma_1(t), \gamma_2(2), \gamma_3(t)$ の形である
シェイプ作用素は負の法ベクトルの t に関する導関数であるから，$\mathbf{n}(\gamma(t))$ を t に関して微分すると

$$\sum_{\mathbf{p}} \mathbf{v} = -\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}t}(\gamma(t))\bigg|_{t=0} \tag{100}$$

$$\overset{t=0\text{は放っておいて}}{=} \begin{pmatrix} \frac{\gamma_1^2\gamma_1'\sqrt{\gamma_1^2+\gamma_2^2}+\gamma_2^2\gamma_1'\left(\sqrt{\gamma_1^2+\gamma_2^2}-R\right)+R\gamma_1\gamma_2\gamma_2'}{r\left(\gamma_1^2+\gamma_2^2\right)^{\frac{3}{2}}} \\ \frac{\gamma_2^2\gamma_2'\sqrt{\gamma_1^2+\gamma_2^2}+\gamma_1^2\gamma_2'\left(\sqrt{\gamma_1^2+\gamma_2^2}-R\right)+R\gamma_1\gamma_2\gamma_1'}{r\left(\gamma_1^2+\gamma_2^2\right)^{\frac{3}{2}}} \\ \frac{\gamma_3'}{r} \end{pmatrix} \tag{101}$$

$$\overset{t=0\text{を代入}}{=} \frac{1}{r\left(x^2+y^2\right)^{\frac{3}{2}}} \begin{pmatrix} x^2v_1\sqrt{x^2+y^2}+y^2v_1\left(\sqrt{x^2+y^2}-R\right)+Rxyv_2 \\ y^2v_2\sqrt{x^2+y^2}+x^2v_2\left(\sqrt{x^2+y^2}-R\right)+Rxyv_1 \\ v_3\left(x^2+y^2\right)^{\frac{3}{2}} \end{pmatrix} \tag{102}$$

以下，接空間の基底を考える．$T_{R,r}$ は二つの円環からなるものであるから，一つの基底を $\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$ である（$T_{R,r}$ の赤道面の接ベクトル）．二つ目の基底は必ず法ベクトルと直交するので，$\mathbf{n} \times \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$ を計算すればいい（$\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \times \mathbf{n}$ はただの逆方向であるから外積の順序が逆にしても OK，なお，基底は向きだけ影響するから，長さは単位でなくても OK）

$$\mathbf{n} \times \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} \left(\sqrt{x^2+y^2}-R\right)\frac{x}{\sqrt{x^2+y^2}} \\ \left(\sqrt{x^2+y^2}-R\right)\frac{y}{\sqrt{x^2+y^2}} \\ z \end{pmatrix} \times \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \tag{103}$$

$$= \begin{pmatrix} -xz \\ -yz \\ \left(\sqrt{x^2+y^2}-R\right)\sqrt{x^2+y^2} \end{pmatrix} \tag{104}$$

よって，接空間の基底は $\mathbf{e}_1 = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} -xz \\ -yz \\ \left(\sqrt{x^2+y^2}-R\right)\sqrt{x^2+y^2} \end{pmatrix}$ である

すると, $\sum_{\mathbf{p}} \mathbf{v} = \frac{1}{r}\left(1-\frac{R}{\sqrt{x^2+y^2}}\right)\mathbf{e}_1 + \frac{1}{r}\mathbf{e}_2$ となり, 表現行列は $P = \begin{pmatrix} \frac{1}{r}\left(1-\frac{R}{\sqrt{x^2+y^2}}\right) & 0 \\ 0 & \frac{1}{r} \end{pmatrix}$ になる

よって

$$\begin{cases} \kappa_1 = \frac{1}{r}\left(1-\frac{R}{\sqrt{x^2+y^2}}\right) \\ \kappa_2 = \dfrac{1}{r} \\ K = \det P = \kappa_1\kappa_2 = \frac{1}{r^2}\left(1-\frac{R}{\sqrt{x^2+y^2}}\right) \\ H = \frac{1}{2}\mathrm{tr}P = \frac{\kappa_1+\kappa_2}{2} = \frac{1}{2r}\left(2-\frac{R}{\sqrt{x^2+y^2}}\right) \end{cases}$$

3.4 E3.4

$\gamma(0) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \frac{\mathrm{d}\gamma}{\mathrm{d}t}(0) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ をみたす $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$ とすると

$$\sum_{\mathbf{p}} \mathbf{v} = -\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}t}(\gamma(t))\Big|_{t=0} \tag{105}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{r}\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ 0 \end{pmatrix}\right) \tag{106}$$

$$= -\frac{1}{r}\begin{pmatrix} \gamma_1' \\ \gamma_2' \\ 0 \end{pmatrix} = -\frac{1}{r}\begin{pmatrix} \gamma_1' \\ \gamma_2' \\ \gamma_3' - \gamma_3' \end{pmatrix} \tag{107}$$

$$= -\frac{1}{r}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \frac{1}{r}\begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} = -\frac{1}{r}\mathbf{v} + \frac{\mathbf{v}\cdot\mathbf{e}_3}{r}\mathbf{e}_3 \tag{108}$$

3.5 P3.1

$f=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1, \nabla f=\begin{pmatrix}\frac{2x}{a^2}\\ \frac{2y}{b^2}\\ \frac{2z}{c^2}\end{pmatrix}, \|\nabla f\|=2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$

$\frac{\nabla f}{\|\nabla f\|}=\frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}\begin{pmatrix}\frac{x}{a^2}\\ \frac{y}{b^2}\\ \frac{z}{c^2}\end{pmatrix}$

$\sigma_u\times\sigma_v=\begin{pmatrix}bc\sin^2 u\cos v\\ ac\sin^2 u\sin v\\ ab\sin u\cos u\end{pmatrix}$

$\|\sigma_u\times\sigma_v\|=\sqrt{b^2c^2\sin^4 u\cos^2 v+a^2c^2\sin^4 u\sin^2 v+a^2b^2\sin^2 u\cos^2 u}$

$\sigma_u=\begin{pmatrix}a\cos u\cos v\\ b\cos u\sin v\\ -c\sin u\end{pmatrix}, \sigma_v=\begin{pmatrix}-a\sin u\sin v\\ b\sin u\cos v\\ 0\end{pmatrix}$

$\begin{cases}E=\sigma_u\cdot\sigma_u=a^2\cos^2 u\cos^2 v+b^2\cos^2 u\sin^2 v+c\sin^2 u\\ F=\sigma_u\cdot\sigma_v=\left(b^2-a^2\right)\sin u\cos u\sin v\cos v\\ G=\sigma_v\cdot\sigma_v=a^2\sin^2 u\sin^2 v+b^2\sin^2 u\cos^2 v\end{cases}$

$\mathbf{n}=\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}=\frac{1}{\sqrt{b^2c^2\sin^4 u\cos^2 v+a^2c^2\sin^4 u\sin^2 v+a^2b^2\sin^2 u\cos^2 u}}\begin{pmatrix}bc\sin^2 u\cos v\\ ac\sin^2 u\sin v\\ ab\sin u\cos u\end{pmatrix}$

$\sigma_{uu}=\begin{pmatrix}-a\sin u\cos v\\ -b\sin u\sin v\\ -c\cos u\end{pmatrix}, \sigma_{uv}=\begin{pmatrix}-a\cos u\sin v\\ b\cos u\cos v\\ 0\end{pmatrix}, \sigma_{vv}=\begin{pmatrix}-a\sin u\cos v\\ -b\sin u\sin v\\ 0\end{pmatrix}$

$$L=\sigma_{uu}\cdot\mathbf{n} \tag{109}$$

$$=\frac{1}{\sqrt{b^2c^2\sin^4 u\cos^2 v+a^2c^2\sin^4 u\sin^2 v+a^2b^2\sin^2 u\cos^2 u}}\begin{pmatrix}-a\sin u\cos v\\ -b\sin u\sin v\\ -c\cos u\end{pmatrix}\cdot\begin{pmatrix}bc\sin^2 u\cos v\\ ac\sin^2 u\sin v\\ ab\sin u\cos u\end{pmatrix} \tag{110}$$

$$=-\frac{abc\sin u}{\sqrt{b^2c^2\sin^4 u\cos^2 v+a^2c^2\sin^4 u\sin^2 v+a^2b^2\sin^2 u\cos^2 u}}\left(\sin^2 u+\cos^2 u\right) \tag{111}$$

$$=-\frac{abc\sin u}{\sqrt{b^2c^2\sin^4 u\cos^2 v+a^2c^2\sin^4 u\sin^2 v+a^2b^2\sin^2 u\cos^2 u}} \tag{112}$$

$$M=\sigma_{uv}\cdot\mathbf{n} \tag{113}$$

$$=\frac{1}{\sqrt{b^2c^2\sin^4 u\cos^2 v+a^2c^2\sin^4 u\sin^2 v+a^2b^2\sin^2 u\cos^2 u}}\begin{pmatrix}-a\cos u\sin v\\ b\cos u\cos v\\ 0\end{pmatrix}\cdot\begin{pmatrix}bc\sin^2 u\cos v\\ ac\sin^2 u\sin v\\ ab\sin u\cos u\end{pmatrix} \tag{114}$$

$$=0 \tag{115}$$

$$N = \sigma_{vv} \cdot \mathbf{n} \tag{116}$$

$$= \frac{1}{\sqrt{b^2c^2\sin^4 u\cos^2 v + a^2c^2\sin^4 u\sin^2 v + a^2b^2\sin^2 u\cos^2 u}} \begin{pmatrix} -a\sin u\cos v \\ -b\sin u\sin v \\ 0 \end{pmatrix} \cdot \begin{pmatrix} bc\sin^2 u\cos v \\ ac\sin^2 u\sin v \\ ab\sin u\cos u \end{pmatrix} \tag{117}$$

$$= -\frac{abc\sin^3 u}{\sqrt{b^2c^2\sin^4 u\cos^2 v + a^2c^2\sin^4 u\sin^2 v + a^2b^2\sin^2 u\cos^2 u}} \tag{118}$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{1}{a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)}$$

$$H = \frac{GL - 2FM + EN}{2\left(EG - F^2\right)} = -\frac{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - \frac{\frac{x^2}{a^6} + \frac{y^2}{b^6} + \frac{z^2}{c^6}}{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}{2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}$$

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K} = -\frac{\sum\frac{1}{a^2} - \frac{\sum\frac{x^2}{a^6}}{\sum\frac{x^2}{a^4}}}{2\sqrt{\sum\frac{x^2}{a^4}}} \pm \sqrt{\frac{\left(\sum\frac{bc}{a} - \frac{\sum\frac{x^2bc}{a^5}}{\sum\frac{x^2}{a^4}}\right)^2 - 4}{4\sum\frac{x^2}{a^4}}}$$

3.6 P3.2

$$\sigma_u = \begin{pmatrix} a\sinh u\cos v \\ b\sinh u\sin v \\ c\cosh u \end{pmatrix}, \sigma_v = \begin{pmatrix} -a\cosh u\sin v \\ b\cosh u\cos v \\ 0 \end{pmatrix}$$

$$\begin{cases} E = \sigma_u \cdot \sigma_u = a^2\sinh^2 u\cos^2 v + b^2\sinh^2 u\sin^2 v + c\cosh^2 u \\ F = \sigma_u \cdot \sigma_v = \left(b^2 - a^2\right)\sinh u\cosh u\sin v\cos v \\ G = \sigma_v \cdot \sigma_v = a^2\cosh^2 u\sin^2 v + b^2\cosh^2 u\cos^2 v \end{cases}$$

$$f = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1, \nabla f = \begin{pmatrix} \frac{2x}{a^2} \\ \frac{2y}{b^2} \\ -\frac{2z}{c^2} \end{pmatrix}, \|\nabla f\| = 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} -bc\cosh^2 u\cos v \\ -ac\cosh^2 u\sin v \\ ab\sinh u\cosh u \end{pmatrix}$$

$$\|\sigma_u \times \sigma_v\| = \sqrt{b^2c^2\cosh^4 u\cos^2 v + a^2c^2\cosh^4 u\sin^2 v + a^2b^2\sinh^2 u\cosh^2 u}$$

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{b^2c^2\cosh^4 u\cos^2 v + a^2c^2\cosh^4 u\sin^2 v + a^2b^2\sinh^2 u\cosh^2 u}} \begin{pmatrix} -bc\cosh^2 u\cos v \\ -ac\cosh^2 u\sin v \\ ab\sinh u\cosh u \end{pmatrix}$$

$= -\frac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|}$ は σ が正の向きになる単位法ベクトル場

$$\sigma_{uu}=\begin{pmatrix} a\cosh u\cos v\\ b\cosh u\sin v\\ c\sinh u\end{pmatrix},\sigma_{uv}=\begin{pmatrix} -a\sinh u\sin v\\ b\sinh u\cos v\\ 0\end{pmatrix},\sigma_{vv}=\begin{pmatrix} -a\cosh u\cos v\\ -b\cosh u\sin v\\ 0\end{pmatrix}$$

$$L=\sigma_{uu}\cdot\mathbf{n} \tag{119}$$

$$=\begin{pmatrix} a\cosh u\cos v\\ b\cosh u\sin v\\ c\sinh u\end{pmatrix}\cdot\begin{pmatrix} -bc\cosh^2 u\cos v\\ -ac\cosh^2 u\sin v\\ ab\sinh u\cosh u\end{pmatrix} \tag{120}$$

$$\cdot\frac{1}{\sqrt{b^2c^2\cosh^4 u\cos^2 v+a^2c^2\cosh^4 u\sin^2 v+a^2b^2\sinh^2 u\cosh^2 u}} \tag{121}$$

$$=-\frac{abc\cosh u}{\sqrt{b^2c^2\cosh^4 u\cos^2 v+a^2c^2\cosh^4 u\sin^2 v+a^2b^2\sinh^2 u\cosh^2 u}} \tag{122}$$

$$M=\sigma_{uv}\cdot\mathbf{n} \tag{123}$$

$$=\frac{1}{\sqrt{b^2c^2\cosh^4 u\cos^2 v+a^2c^2\cosh^4 u\sin^2 v+a^2b^2\sinh^2 u\cosh^2 u}} \tag{124}$$

$$\cdot\begin{pmatrix} -a\sinh u\sin v\\ b\sinh u\cos v\\ 0\end{pmatrix}\cdot\begin{pmatrix} -bc\cosh^2 u\cos v\\ -ac\cosh^2 u\sin v\\ ab\sinh u\cosh u\end{pmatrix} \tag{125}$$

$$=0 \tag{126}$$

$$N=\sigma_{vv}\cdot\mathbf{n} \tag{127}$$

$$=\frac{1}{\sqrt{b^2c^2\cosh^4 u\cos^2 v+a^2c^2\cosh^4 u\sin^2 v+a^2b^2\sinh^2 u\cosh^2 u}} \tag{128}$$

$$\cdot\begin{pmatrix} -a\cosh u\cos v\\ -b\cosh u\sin v\\ 0\end{pmatrix}\cdot\begin{pmatrix} -bc\cosh^2 u\cos v\\ -ac\cosh^2 u\sin v\\ ab\sinh u\cosh u\end{pmatrix} \tag{129}$$

$$=\frac{abc\cosh^3 u}{\sqrt{b^2c^2\cosh^4 u\cos^2 v+a^2c^2\cosh^4 u\sin^2 v+a^2b^2\sinh^2 u\cosh^2 u}} \tag{130}$$

$$K=\frac{LN-M^2}{EG-F^2}=-\frac{1}{a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^2}$$

$$H=\frac{GL-2FM+EN}{2\left(EG-F^2\right)}=\frac{1}{2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}\left(\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right)-\frac{\frac{x^2}{a^6}+\frac{y^2}{b^6}+\frac{z^2}{c^6}}{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}\right)$$

$$\kappa_{1,2}=H\pm\sqrt{H^2-K} \tag{131}$$

$$=\frac{1}{2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}\left(\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right)-\frac{\frac{x^2}{a^6}+\frac{y^2}{b^6}+\frac{z^2}{c^6}}{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}\right) \tag{132}$$

$$\pm\sqrt{\frac{1}{4a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^2}\left(\left(\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right)\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}-\frac{\frac{x^2}{a^6}+\frac{y^2}{b^6}+\frac{z^2}{c^6}}{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{3}{2}}}\right)^2+4\right)} \tag{133}$$

3.7 P3.3

$\sigma_+ : \sigma_u = \begin{pmatrix} a \\ 0 \\ \frac{cu}{\sqrt{u^2+v^2+1}} \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ b \\ \frac{cv}{\sqrt{u^2+v^2+1}} \end{pmatrix}, \sigma_- : \sigma_u = \begin{pmatrix} a \\ 0 \\ -\frac{cu}{\sqrt{u^2+v^2+1}} \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ b \\ -\frac{cv}{\sqrt{u^2+v^2+1}} \end{pmatrix}$

$$\begin{cases} E_+ = \sigma_u \cdot \sigma_u = a^2 + \dfrac{c^2u^2}{u^2+v^2+1} \\ F_+ = \sigma_u \cdot \sigma_v = \dfrac{c^2uv}{u^2+v^2+1} \\ G_+ = \sigma_v \cdot \sigma_v = b^2 + \dfrac{c^2v^2}{u^2+v^2+1} \end{cases}, \begin{cases} E_- = \sigma_u \cdot \sigma_u = a^2 + \dfrac{c^2u^2}{u^2+v^2+1} \\ F_- = \sigma_u \cdot \sigma_v = \dfrac{c^2uv}{u^2+v^2+1} \\ G_- = \sigma_v \cdot \sigma_v = b^2 + \dfrac{c^2v^2}{u^2+v^2+1} \end{cases}$$

σ_+ :

$\sigma_u \times \sigma_v = \begin{pmatrix} -\frac{bcu}{\sqrt{u^2+v^2+1}} \\ -\frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \sqrt{\dfrac{b^2c^2u^2}{u^2+v^2+1} + \dfrac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}$

σ_- :

$\sigma_u \times \sigma_v = \begin{pmatrix} \frac{bcu}{\sqrt{u^2+v^2+1}} \\ \frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \sqrt{\dfrac{b^2c^2u^2}{u^2+v^2+1} + \dfrac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}$

$\nabla f = \begin{pmatrix} \frac{2x}{a^2} \\ \frac{2y}{b^2} \\ -\frac{2z}{c^2} \end{pmatrix}, \|\nabla f\| = 2\sqrt{\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4}}$

よって，$\mathbf{n}_+ = \dfrac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}} \begin{pmatrix} -\frac{bcu}{\sqrt{u^2+v^2+1}} \\ -\frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix} = -\dfrac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|}$

$\mathbf{n}_- = \dfrac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}} \begin{pmatrix} \frac{bcu}{\sqrt{u^2+v^2+1}} \\ \frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix} = \dfrac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|}$ はそれぞれ σ_+, σ_- が正の向きになる単位法ベクトル場である

σ_+ :

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ \frac{c\left(v^2+1\right)}{(u^2+v^2+1)^{\frac{3}{2}}} \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} 0 \\ 0 \\ -\frac{cuv}{(u^2+v^2+1)^{\frac{3}{2}}} \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} 0 \\ 0 \\ \frac{c\left(u^2+1\right)}{(u^2+v^2+1)^{\frac{3}{2}}} \end{pmatrix}$$

$$L = \sigma_{uu} \cdot \mathbf{n} \tag{134}$$

$$= \frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}} \begin{pmatrix} 0 \\ 0 \\ \frac{c\left(v^2+1\right)}{(u^2+v^2+1)^{\frac{3}{2}}} \end{pmatrix} \cdot \begin{pmatrix} -\frac{bcu}{\sqrt{u^2+v^2+1}} \\ -\frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix} \tag{135}$$

$$= \frac{abc\left(v^2+1\right)}{(u^2+v^2+1)\sqrt{b^2c^2u^2 + a^2c^2v^2 + a^2b^2\left(u^2+v^2+1\right)}} \tag{136}$$

$$M = \sigma_{uv} \cdot \mathbf{n} \tag{137}$$

$$= \frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}} \begin{pmatrix} 0 \\ 0 \\ -\frac{cuv}{(u^2+v^2+1)^{\frac{3}{2}}} \end{pmatrix} \cdot \begin{pmatrix} -\frac{bcu}{\sqrt{u^2+v^2+1}} \\ -\frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix} \tag{138}$$

$$= -\frac{abcuv}{(u^2+v^2+1)\sqrt{b^2c^2u^2 + a^2c^2v^2 + a^2b^2\left(u^2+v^2+1\right)}} \tag{139}$$

$$N = \sigma_{vv} \cdot \mathbf{n} \tag{140}$$

$$= \frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}} \begin{pmatrix} 0 \\ 0 \\ \frac{c\left(u^2+1\right)}{(u^2+v^2+1)^{\frac{3}{2}}} \end{pmatrix} \cdot \begin{pmatrix} -\frac{bcu}{\sqrt{u^2+v^2+1}} \\ -\frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix} \tag{141}$$

$$= \frac{abc\left(u^2+1\right)}{(u^2+v^2+1)\sqrt{b^2c^2u^2 + a^2c^2v^2 + a^2b^2\left(u^2+v^2+1\right)}} \tag{142}$$

$\sigma_- :$

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ -\frac{c\left(v^2+1\right)}{\left(u^2+v^2+1\right)^{\frac{3}{2}}} \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} 0 \\ 0 \\ \frac{cuv}{\left(u^2+v^2+1\right)^{\frac{3}{2}}} \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} 0 \\ 0 \\ -\frac{c\left(u^2+1\right)}{\left(u^2+v^2+1\right)^{\frac{3}{2}}} \end{pmatrix}$$

$$L = \sigma_{uu} \cdot \mathbf{n} \tag{143}$$

$$= \frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}} \begin{pmatrix} 0 \\ 0 \\ -\frac{c\left(v^2+1\right)}{\left(u^2+v^2+1\right)^{\frac{3}{2}}} \end{pmatrix} \cdot \begin{pmatrix} \frac{bcu}{\sqrt{u^2+v^2+1}} \\ \frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix} \tag{144}$$

$$= -\frac{abc\left(v^2+1\right)}{\left(u^2+v^2+1\right)\sqrt{b^2c^2u^2 + a^2c^2v^2 + a^2b^2\left(u^2+v^2+1\right)}} \tag{145}$$

$$M = \sigma_{uv} \cdot \mathbf{n} \tag{146}$$

$$= \frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}} \begin{pmatrix} 0 \\ 0 \\ \frac{cuv}{\left(u^2+v^2+1\right)^{\frac{3}{2}}} \end{pmatrix} \cdot \begin{pmatrix} \frac{bcu}{\sqrt{u^2+v^2+1}} \\ \frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix} \tag{147}$$

$$= \frac{abcuv}{\left(u^2+v^2+1\right)\sqrt{b^2c^2u^2 + a^2c^2v^2 + a^2b^2\left(u^2+v^2+1\right)}} \tag{148}$$

$$N = \sigma_{vv} \cdot \mathbf{n} \tag{149}$$

$$= \frac{1}{\sqrt{\frac{b^2c^2u^2}{u^2+v^2+1} + \frac{a^2c^2v^2}{u^2+v^2+1} + a^2b^2}} \begin{pmatrix} 0 \\ 0 \\ -\frac{c\left(u^2+1\right)}{\left(u^2+v^2+1\right)^{\frac{3}{2}}} \end{pmatrix} \cdot \begin{pmatrix} \frac{bcu}{\sqrt{u^2+v^2+1}} \\ \frac{acv}{\sqrt{u^2+v^2+1}} \\ ab \end{pmatrix} \tag{150}$$

$$= -\frac{abc\left(u^2+1\right)}{\left(u^2+v^2+1\right)\sqrt{b^2c^2u^2 + a^2c^2v^2 + a^2b^2\left(u^2+v^2+1\right)}} \tag{151}$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{1}{a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^2}$$

$$H = \frac{GL - 2FM + EN}{2\left(EG - F^2\right)} = \pm\frac{1}{2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}\left(\left(\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2}\right) - \frac{1}{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}\left(\frac{x^2}{a^6} + \frac{y^2}{b^6} - \frac{z^2}{c^6}\right)\right)$$

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K} \tag{152}$$

$$= \pm\frac{1}{2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}\left(\left(\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2}\right) - \frac{1}{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}\left(\frac{x^2}{a^6} + \frac{y^2}{b^6} - \frac{z^2}{c^6}\right)\right) \tag{153}$$

$$\mp \sqrt{\frac{1}{4a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^2}\left(a^2b^2c^2\left(\left(\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2}\right) - \frac{\frac{x^2}{a^6} + \frac{y^2}{b^6} - \frac{z^2}{c^6}}{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}\right)^2 + 4\right)} \tag{154}$$

3.8 P3.4

$$\sigma_u = \begin{pmatrix} 1 \\ 0 \\ \frac{2u}{a^2} \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 1 \\ \frac{2v}{b^2} \end{pmatrix}, \begin{cases} E = \sigma_u \cdot \sigma_u = 1 + \dfrac{4u^2}{a^4} \\ F = \sigma_u \cdot \sigma_v = \dfrac{4uv}{a^2 b^2} \\ G = \sigma_v \cdot \sigma_v = 1 + \dfrac{4v^2}{b^4} \end{cases}$$

$$f = \frac{x^2}{a^2} + \frac{y^2}{b^2} - z, \nabla f = \begin{pmatrix} \frac{2x}{a^2} \\ \frac{2y}{b^2} \\ -1 \end{pmatrix}$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} -\frac{2u}{a^2} \\ -\frac{2v}{b^2} \\ 1 \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}$$

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} -\frac{2u}{a^2} \\ -\frac{2v}{b^2} \\ 1 \end{pmatrix} = -\frac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|}$$

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{a^2} \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{b^2} \end{pmatrix}$$

$$L = \sigma_{uu} \cdot \mathbf{n} \tag{155}$$

$$= \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} 0 \\ 0 \\ \frac{2}{a^2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{2u}{a^2} \\ -\frac{2v}{b^2} \\ 1 \end{pmatrix} \tag{156}$$

$$= \frac{2}{a^2 \sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \tag{157}$$

$$M = \sigma_{uv} \cdot \mathbf{n} \tag{158}$$

$$= \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\frac{2u}{a^2} \\ -\frac{2v}{b^2} \\ 1 \end{pmatrix} \tag{159}$$

$$= 0 \tag{160}$$

$$N = \sigma_{vv} \cdot \mathbf{n} \tag{161}$$

$$= \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} 0 \\ 0 \\ \frac{2}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{2u}{a^2} \\ -\frac{2v}{b^2} \\ 1 \end{pmatrix} \tag{162}$$

$$= \frac{2}{b^2 \sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \tag{163}$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{4}{a^2 b^2 \left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^2}$$

$$H = \frac{GL - 2FM + EN}{2\left(EG - F^2\right)} = \frac{1}{\left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^{\frac{3}{2}}} \left(\frac{1}{a^2}\left(\frac{4v^2}{b^4} + 1\right) + \frac{1}{b^2}\left(\frac{4u^2}{a^4} + 1\right)\right)$$

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K} \tag{164}$$

$$= \frac{1}{\left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^{\frac{3}{2}}} \left(\frac{1}{a^2}\left(\frac{4v^2}{b^4} + 1\right) + \frac{1}{b^2}\left(\frac{4u^2}{a^4} + 1\right)\right) \tag{165}$$

$$\pm \sqrt{\frac{1}{a^2 b^2 \left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^3} \left(\left(\frac{b}{a}\left(\frac{4v^2}{b^4} + 1\right) + \frac{a}{b}\left(\frac{4u^2}{a^4} + 1\right)\right)^2 - \left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)\right)} \tag{166}$$

3.9 P3.5

$$\sigma_u = \begin{pmatrix} 1 \\ 0 \\ \frac{2u}{a^2} \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 1 \\ -\frac{2v}{b^2} \end{pmatrix}, \begin{cases} E = \sigma_u \cdot \sigma_u = 1 + \frac{4u^2}{a^4} \\ F = \sigma_u \cdot \sigma_v = -\frac{4uv}{a^2 b^2} \\ G = \sigma_v \cdot \sigma_v = 1 + \frac{4v^2}{b^4} \end{cases}$$

$$f = \frac{x^2}{a^2} - \frac{y^2}{b^2} - z, \nabla f = \begin{pmatrix} \frac{2x}{a^2} \\ -\frac{2y}{b^2} \\ -1 \end{pmatrix}, \|\nabla f\| = \sqrt{\frac{4x^2}{a^4} + \frac{4y^2}{b^4} + 1}$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} -\frac{2u}{a^2} \\ \frac{2v}{b^2} \\ 1 \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}$$

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} -\frac{2u}{a^2} \\ \frac{2v}{b^2} \\ 1 \end{pmatrix} = -\frac{\left(\nabla f\right)\left(\sigma\left(u, v\right)\right)}{\left\|\left(\nabla f\right)\left(\sigma\left(u, v\right)\right)\right\|}$$

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{a^2} \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} 0 \\ 0 \\ -\frac{2}{b^2} \end{pmatrix}$$

$$L = \sigma_{uu} \cdot \mathbf{n} \tag{167}$$
$$= \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} 0 \\ 0 \\ \frac{2}{a^2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{2u}{a^2} \\ \frac{2v}{b^2} \\ 1 \end{pmatrix} \tag{168}$$
$$= \frac{2}{a^2\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \tag{169}$$
$$M = \sigma_{uv} \cdot \mathbf{n} \tag{170}$$
$$= \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\frac{2u}{a^2} \\ \frac{2v}{b^2} \\ 1 \end{pmatrix} \tag{171}$$
$$= 0 \tag{172}$$
$$N = \sigma_{vv} \cdot \mathbf{n} \tag{173}$$
$$= \frac{1}{\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \begin{pmatrix} 0 \\ 0 \\ -\frac{2}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{2u}{a^2} \\ \frac{2v}{b^2} \\ 1 \end{pmatrix} \tag{174}$$
$$= -\frac{2}{b^2\sqrt{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1}} \tag{175}$$

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{4}{a^2b^2\left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^2}$$

$$H = \frac{GL - 2FM + EN}{2\left(EG - F^2\right)} = \frac{1}{\left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^{\frac{3}{2}}} \left(\frac{1}{a^2}\left(\frac{4v^2}{b^4} + 1\right) - \frac{1}{b^2}\left(\frac{4u^2}{a^4} + 1\right)\right)$$

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K} \tag{176}$$
$$= \frac{1}{\left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^{\frac{3}{2}}} \left(\frac{1}{a^2}\left(\frac{4v^2}{b^4} + 1\right) - \frac{1}{b^2}\left(\frac{4u^2}{a^4} + 1\right)\right) \tag{177}$$
$$\pm \sqrt{\frac{1}{a^2b^2\left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^3}\left(\left(\frac{b}{a}\left(\frac{4v^2}{b^4} + 1\right) - \frac{a}{b}\left(\frac{4u^2}{a^4} + 1\right)\right)^2 + 4\left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)\right)} \tag{178}$$

3.10 P3.6

$$\sigma_u = \begin{pmatrix} 1 \\ 3u^2 \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{cases} E = \sigma_u \cdot \sigma_u = 9u^4 + 1 \\ F = \sigma_u \cdot \sigma_v = 0 \\ G = \sigma_v \cdot \sigma_v = 1 \end{cases}$$

$$f = x^3 - y, \nabla f = \begin{pmatrix} 3x^2 \\ -1 \\ 0 \end{pmatrix}, \|\nabla f\| = \sqrt{9x^4 + 1}$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} 3u^2 \\ -1 \\ 0 \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \sqrt{9u^4 + 1}$$

$$\mathbf{n}=\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}=\frac{1}{\sqrt{9u^4+1}}\begin{pmatrix}3u^2\\-1\\0\end{pmatrix}=\frac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|}$$

$$\sigma_{uu}=\begin{pmatrix}0\\6u\\0\end{pmatrix},\sigma_{uv}=\begin{pmatrix}0\\0\\0\end{pmatrix},\sigma_{vv}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{aligned}
L&=\sigma_{uu}\cdot\mathbf{n} && (179)\\
&=\frac{1}{\sqrt{9u^4+1}}\begin{pmatrix}0\\6u\\0\end{pmatrix}\cdot\begin{pmatrix}3u^2\\-1\\0\end{pmatrix} && (180)\\
&=-\frac{6u}{\sqrt{9u^4+1}} && (181)\\
M&=\sigma_{uv}\cdot\mathbf{n} && (182)\\
&=\frac{1}{\sqrt{9u^4+1}}\begin{pmatrix}0\\0\\0\end{pmatrix}\cdot\begin{pmatrix}3u^2\\-1\\0\end{pmatrix} && (183)\\
&=0 && (184)\\
N&=\sigma_{vv}\cdot\mathbf{n} && (185)\\
&=\frac{1}{\sqrt{9u^4+1}}\begin{pmatrix}0\\0\\0\end{pmatrix}\cdot\begin{pmatrix}3u^2\\-1\\0\end{pmatrix} && (186)\\
&=0 && (187)
\end{aligned}$$

$$K=\frac{LN-M^2}{EG-F^2}=0$$
$$H=\frac{GL-2FM+EN}{2(EG-F^2)}=-\frac{3u}{(9u^4+1)^{\frac{3}{2}}}$$

$$\begin{aligned}
\kappa_{1,2}&=H\pm\sqrt{H^2-K} && (188)\\
&=-\frac{3u}{(9u^4+1)^{\frac{3}{2}}}\pm\frac{3u}{(9u^4+1)^{\frac{3}{2}}} && (189)
\end{aligned}$$

3.11 P3.7

$$\sigma_u=\begin{pmatrix}1\\0\\\tan u\end{pmatrix},\sigma_v=\begin{pmatrix}0\\1\\-\tan v\end{pmatrix},\begin{cases}E=\sigma_u\cdot\sigma_u=\dfrac{1}{\cos^2 u}\\F=\sigma_u\cdot\sigma_v=-\tan u\tan v\\G=\sigma_v\cdot\sigma_v=\dfrac{1}{\cos^2 v}\end{cases}$$

(2)

$$f=e^z\cos x-\cos y,\nabla f=\begin{pmatrix}-e^z\sin x\\\sin y\\e^z\cos x\end{pmatrix},\|\nabla f\|=\sqrt{e^{2z}+\sin^2 y}$$

$$\sigma_u\times\sigma_v=\begin{pmatrix}-\tan u\\\tan v\\1\end{pmatrix},\|\sigma_u\times\sigma_v\|=\sqrt{\tan^2 u+\tan^2 v+1}$$

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{\tan^2 u + \tan^2 v + 1}} \begin{pmatrix} -\tan u \\ \tan v \\ 1 \end{pmatrix} = \frac{(\nabla f)(\sigma(u,v))}{\|(\nabla f)(\sigma(u,v))\|}$$

(3)

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\cos^2 u} \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\cos^2 v} \end{pmatrix}$$

$$L = \sigma_{uu} \cdot \mathbf{n} \tag{190}$$
$$= \frac{1}{\sqrt{\tan^2 u + \tan^2 v + 1}} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\cos^2 u} \end{pmatrix} \cdot \begin{pmatrix} -\tan u \\ \tan v \\ 1 \end{pmatrix} \tag{191}$$
$$= \frac{1}{\cos^2 u \sqrt{\tan^2 u + \tan^2 v + 1}} \tag{192}$$
$$M = \sigma_{uv} \cdot \mathbf{n} \tag{193}$$
$$= \frac{1}{\sqrt{\tan^2 u + \tan^2 v + 1}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\tan u \\ \tan v \\ 1 \end{pmatrix} \tag{194}$$
$$= 0 \tag{195}$$
$$N = \sigma_{vv} \cdot \mathbf{n} \tag{196}$$
$$= \frac{1}{\sqrt{\tan^2 u + \tan^2 v + 1}} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\cos^2 v} \end{pmatrix} \cdot \begin{pmatrix} -\tan u \\ \tan v \\ 1 \end{pmatrix} \tag{197}$$
$$= -\frac{1}{\cos^2 v \sqrt{\tan^2 u + \tan^2 v + 1}} \tag{198}$$

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{1}{(\tan^2 u + \tan^2 v + 1)\left(1 - \sin^2 u \sin^2 v\right)}$$
$$H = \frac{GL - 2FM + EN}{2(EG - F^2)} = 0$$

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K} \tag{199}$$
$$= \pm \sqrt{\frac{1}{(\tan^2 u + \tan^2 v + 1)\left(1 - \sin^2 u \sin^2 v\right)}} \tag{200}$$

4 §4

4.1 E4.1

$$\sigma_u = \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} -u\sin v \\ u\cos v \\ k \end{pmatrix}, \begin{cases} E = \sigma_u \cdot \sigma_u = 1 \\ F = \sigma_u \cdot \sigma_v = 0 \\ G = \sigma_v \cdot \sigma_v = u^2 + k^2 \end{cases}$$

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{u^2+k^2}} \begin{pmatrix} k\sin v \\ -k\cos v \\ u \end{pmatrix}$$

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} -\sin v \\ \cos v \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} -u\cos v \\ -u\sin v \\ 0 \end{pmatrix}, \begin{cases} L = \sigma_{uu} \cdot \mathbf{n} = 0 \\ M = \sigma_{uv} \cdot \mathbf{n} = -\frac{k}{\sqrt{u^2+k^2}} \\ N = 0 \end{cases}$$

$$K = \frac{LN - M^2}{EG - F^2} \tag{201}$$

$$= -\frac{k^2}{(u^2+k^2)^2} \tag{202}$$

$$H = \frac{GL - 2FM + EN}{2(EG - F^2)} = 0 \tag{203}$$

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K} = \pm\frac{k}{u^2+k^2} \tag{204}$$

$-\frac{k^2}{(u^2+k^2)^2} < 0$ であるから（$k > 0$ より），楕円点と放物点全体の集合は空集合で，全ての点は双曲点である

一方，$\kappa_1 = \kappa_2$ とすると，$k = 0$ になるから，臍点全体の集合も空集合である

4.2 E4.2

$$K(\mathbf{p}) = \frac{1}{r^2}\left(1 - \frac{R}{\sqrt{x^2+y^2}}\right), \kappa_1 = \frac{1}{r}\left(1 - \frac{R}{\sqrt{x^2+y^2}}\right), \kappa_2 = \frac{1}{r}$$ であるから

$$\begin{cases} K > 0 \iff \sqrt{x^2+y^2} > R \\ K = 0 \iff \sqrt{x^2+y^2} = R \iff z = \pm r \\ K < 0 \iff \sqrt{x^2+y^2} < R \\ \kappa_1 < \kappa_2 \end{cases}$$ から，$$\begin{cases} \text{楕円点} : \left\{\mathbf{p} \in T_{R,r} : \sqrt{x^2+y^2} > R\right\} \\ \text{放物点} : \{\mathbf{p} \in T_{R,r} : z^2 = r^2\} \\ \text{双曲点} : \left\{\mathbf{p} \in T_{R,r} : \sqrt{x^2+y^2} < R\right\} \\ \text{臍点} : \emptyset \end{cases}$$

4.3 E4.3

$$\sigma_u = \begin{pmatrix} 1 \\ 0 \\ au \end{pmatrix}, \sigma_v = \begin{pmatrix} 0 \\ 1 \\ bv \end{pmatrix}, \begin{cases} E = \sigma_u \cdot \sigma_u = 1 + a^2u^2 \\ F = \sigma_u \cdot \sigma_v = abuv \\ G = \sigma_v \cdot \sigma_v = 1 + b^2v^2 \end{cases}$$

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{1 + a^2u^2 + b^2v^2} \begin{pmatrix} -au \\ -bv \\ 1 \end{pmatrix}$$

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}, \begin{cases} L = \sigma_{uu} \cdot \mathbf{n} = \frac{a}{\sqrt{1+a^2u^2+b^2v^2}} \\ M = \sigma_{uv} \cdot \mathbf{n} = 0 \\ N = \sigma_{vv} \cdot \mathbf{n} = \frac{b}{\sqrt{1+a^2u^2+b^2v^2}} \end{cases}$$

$$K(\mathbf{0}) = \left.\frac{LN - M^2}{EG - F^2}\right|_{(0,0)} = ab \tag{205}$$

$$H(\mathbf{0}) = \left.\frac{GL - 2FM + EN}{2(EG - F^2)}\right|_{(0,0)} = \frac{a+b}{2} \tag{206}$$

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K} = \frac{a+b}{2} \pm \left|\frac{a-b}{2}\right| \tag{207}$$

よって，楕円点になる条件は $ab > 0$ で，臍点になる条件は $a = b$

4.4 P4.1

$$K = \frac{LN - M^2}{EG - F^2} = \frac{1}{a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)}$$

$$H = \frac{GL - 2FM + EN}{2(EG - F^2)} = -\frac{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - \frac{\frac{x^2}{a^6} + \frac{y^2}{b^6} + \frac{z^2}{c^6}}{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}{2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}$$

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K} = -\frac{\sum \frac{1}{a^2} - \frac{\sum \frac{x^2}{a^6}}{\sum \frac{x^2}{a^4}}}{2\sqrt{\sum \frac{x^2}{a^4}}} \pm \sqrt{\frac{\left(\sum \frac{bc}{a} - \frac{\sum \frac{x^2bc}{a^5}}{\sum \frac{x^2}{a^4}}\right)^2 - 4}{4\sum \frac{x^2}{a^4}}}$$

よって，全ての点は楕円点となり，放物点と双曲点からなる集合は $\emptyset$．また，臍点となる条件は $\left(\sum \frac{bc}{a} - \frac{\sum \frac{x^2bc}{a^5}}{\sum \frac{x^2}{a^4}}\right)^2 = 4$

4.5 P4.2

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{1}{a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^2}$$

$$H = \frac{GL - 2FM + EN}{2(EG - F^2)} = \frac{1}{2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}\left(\left(\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2}\right) - \frac{\frac{x^2}{a^6} + \frac{y^2}{b^6} + \frac{z^2}{c^6}}{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}\right)$$

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K} \tag{208}$$

$$= \frac{1}{2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}\left(\left(\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2}\right) - \frac{\frac{x^2}{a^6} + \frac{y^2}{b^6} + \frac{z^2}{c^6}}{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}\right) \tag{209}$$

$$\pm \sqrt{\frac{1}{4a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^2}\left(\left(\left(\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2}\right)\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}} - \frac{\frac{x^2}{a^6} + \frac{y^2}{b^6} + \frac{z^2}{c^6}}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{3}{2}}}\right)^2 + 4\right)} \tag{210}$$

$K < 0$ から全ての点は双曲点であり，楕円点と放物点からなる集合は $\emptyset$．なお，臍点は存在しない

4.6 P4.3

$$K=\frac{LN-M^2}{EG-F^2}=\frac{1}{a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^2}$$

$$H=\frac{GL-2FM+EN}{2\left(EG-F^2\right)}=\pm\frac{1}{2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}\left(\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right)-\frac{1}{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}\left(\frac{x^2}{a^6}+\frac{y^2}{b^6}-\frac{z^2}{c^6}\right)\right)$$

$$\kappa_{1,2}=H\pm\sqrt{H^2-K} \tag{211}$$

$$=\pm\frac{1}{2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}\left(\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right)-\frac{1}{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}\left(\frac{x^2}{a^6}+\frac{y^2}{b^6}-\frac{z^2}{c^6}\right)\right) \tag{212}$$

$$\mp\sqrt{\frac{1}{4a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^2}\left(a^2b^2c^2\left(\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right)-\frac{\frac{x^2}{a^6}+\frac{y^2}{b^6}-\frac{z^2}{c^6}}{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}\right)^2+4\right)} \tag{213}$$

よって，全ての点は楕円点となり，双曲点と放物点からなる集合は $\emptyset$ である．なお，臍点は存在しない

4.7 P4.4

$$K=\frac{LN-M^2}{EG-F^2}=\frac{4}{a^2b^2\left(\frac{4u^2}{a^4}+\frac{4v^2}{b^4}+1\right)^2}$$

$$H=\frac{GL-2FM+EN}{2\left(EG-F^2\right)}=\frac{1}{\left(\frac{4u^2}{a^4}+\frac{4v^2}{b^4}+1\right)^{\frac{3}{2}}}\left(\frac{1}{a^2}\left(\frac{4v^2}{b^4}+1\right)+\frac{1}{b^2}\left(\frac{4u^2}{a^4}+1\right)\right)$$

$$\kappa_{1,2}=H\pm\sqrt{H^2-K} \tag{214}$$

$$=\frac{1}{\left(\frac{4u^2}{a^4}+\frac{4v^2}{b^4}+1\right)^{\frac{3}{2}}}\left(\frac{1}{a^2}\left(\frac{4v^2}{b^4}+1\right)+\frac{1}{b^2}\left(\frac{4u^2}{a^4}+1\right)\right) \tag{215}$$

$$\pm\sqrt{\frac{1}{a^2b^2\left(\frac{4u^2}{a^4}+\frac{4v^2}{b^4}+1\right)^3}\left(\left(\frac{b}{a}\left(\frac{4v^2}{b^4}+1\right)+\frac{a}{b}\left(\frac{4u^2}{a^4}+1\right)\right)^2-\left(\frac{4u^2}{a^4}+\frac{4v^2}{b^4}+1\right)\right)} \tag{216}$$

よって，全ての点は楕円点となり，双曲点と放物点からなる集合は $\emptyset$ である．なお，臍点となる条件は $\left(\frac{b}{a}\left(\frac{4v^2}{b^4}+1\right)+\frac{a}{b}\left(\frac{4u^2}{a^4}+1\right)\right)^2=\frac{4u^2}{a^4}+\frac{4v^2}{b^4}+1$ である

4.8 P4.5

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{4}{a^2 b^2 \left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^2}$$
$$H = \frac{GL - 2FM + EN}{2(EG - F^2)} = \frac{1}{\left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^{\frac{3}{2}}} \left(\frac{1}{a^2}\left(\frac{4v^2}{b^4} + 1\right) - \frac{1}{b^2}\left(\frac{4u^2}{a^4} + 1\right)\right)$$

$$\begin{aligned}
\kappa_{1,2} &= H \pm \sqrt{H^2 - K} && (217)\\
&= \frac{1}{\left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^{\frac{3}{2}}} \left(\frac{1}{a^2}\left(\frac{4v^2}{b^4} + 1\right) - \frac{1}{b^2}\left(\frac{4u^2}{a^4} + 1\right)\right) && (218)\\
&\pm \sqrt{\frac{1}{a^2 b^2 \left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^3} \left(\left(\frac{b}{a}\left(\frac{4v^2}{b^4} + 1\right) - \frac{a}{b}\left(\frac{4u^2}{a^4} + 1\right)\right)^2 + 4\left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)\right)} && (219)
\end{aligned}$$

よって，全ての点は双曲点となり，放物点と双曲点は存在しない．なお，臍点も存在しない

4.9 P4.6

$$K = \frac{LN - M^2}{EG - F^2} = 0$$
$$H = \frac{GL - 2FM + EN}{2(EG - F^2)} = -\frac{3u}{(9u^4 + 1)^{\frac{3}{2}}}$$

$$\begin{aligned}
\kappa_{1,2} &= H \pm \sqrt{H^2 - K} && (220)\\
&= -\frac{3u}{(9u^4 + 1)^{\frac{3}{2}}} \pm \frac{3u}{(9u^4 + 1)^{\frac{3}{2}}} && (221)
\end{aligned}$$

よって，全ての点が放物点となり，楕円点と双曲点は存在しない．なお，臍点となる条件は $u = 0$ である

4.10 P4.7

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{1}{(\tan^2 u + \tan^2 v + 1)\left(1 - \sin^2 u \sin^2 v\right)}$$
$$H = \frac{GL - 2FM + EN}{2(EG - F^2)} = 0$$

$$\begin{aligned}
\kappa_{1,2} &= H \pm \sqrt{H^2 - K} && (222)\\
&= \pm\sqrt{\frac{1}{(\tan^2 u + \tan^2 v + 1)\left(1 - \sin^2 u \sin^2 v\right)}} && (223)
\end{aligned}$$

よって，放物点，楕円点と臍点は存在しないが，全ての点は双曲点となる

5 §5

5.1 E5.1

$\sigma_u = \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} -u\sin v \\ u\cos v \\ k \end{pmatrix}$ であるから，$\begin{cases} E = \sigma_u \cdot \sigma_u = 1 \\ F = \sigma_u \cdot \sigma_v = 0 \\ G = \sigma_v \cdot \sigma_v = u^2 + k^2 \end{cases}$

$\mathbf{I_q}(\xi) = E\xi_1^2 + 2F\xi_1\xi_2 + G\xi_2^2 = \xi_1^2 + \left(u^2 + k^2\right)\xi_2^2$

$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} -\sin v \\ \cos v \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} -u\cos v \\ -u\sin v \\ 0 \end{pmatrix}$ であるから，$\begin{cases} L = \sigma_{uu} \cdot \mathbf{n} = 0 \\ M = \sigma_{uv} \cdot \mathbf{n} = -\frac{k}{\sqrt{u^2+k^2}} \\ N = 0 \end{cases}$

$\mathbf{II_q}(\xi) = L\xi_1^2 + 2M\xi_1\xi_2 + N\xi_2^2 = -\frac{2k}{\sqrt{u^2+k^2}}\xi_1\xi_2$. よって，$\lambda(\xi) = -\frac{2k\xi_1\xi_2}{\left(\xi_1^2 + \left(u^2+k^2\right)\xi_2^2\right)\sqrt{u^2+k^2}}$

$\xi_2 \neq 0$ のとき，$\lambda(\xi) = -\frac{2k\left(\frac{\xi_1}{\xi_2}\right)}{\left(\left(\frac{\xi_1}{\xi_2}\right)^2 + u^2 + k^2\right)\sqrt{u^2+k^2}}$

ここで，$\alpha : \mathbb{R} \to \mathbb{R}$ を $\alpha(t) = -\frac{2kt}{\left(t^2 + u^2 + k^2\right)\sqrt{u^2+k^2}}$ で定めると

$$\alpha'(t) = \frac{2k\left(t^2 - \left(u^2+k^2\right)\right)}{\left(t^2+u^2+k^2\right)^2\sqrt{u^2+k^2}} \tag{224}$$

$\alpha'(t) = 0$ となるのは $t = \pm\sqrt{u^2+k^2}$ であるから，$\lambda_{max} = \frac{k}{u^2+k^2}, \lambda_{min} = -\frac{k}{u^2+k^2}$ である

5.2 E5.2

$\sigma_u = \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix}, \sigma_v = \begin{pmatrix} -\left(R + r\cos u\right)\sin v \\ \left(R + r\cos u\right)\cos v \\ 0 \end{pmatrix}$ であるから

$\begin{cases} E = \sigma_u \cdot \sigma_u = r^2 \\ F = \sigma_u \cdot \sigma_v = 0 \\ G = \sigma_v \cdot \sigma_v = \left(R + r\cos u\right)^2 \end{cases} \Longrightarrow \mathbf{I_q}(\xi) = E\xi_1^2 + 2F\xi_1\xi_2 + G\xi_2^2 = r^2\xi_1^2 + \left(R + r\cos u\right)^2\xi_2^2$

$\sigma_{uu} = \begin{pmatrix} -r\cos u\cos v \\ -r\cos u\sin v \\ -r\sin u \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} r\sin u\sin v \\ -r\sin u\cos v \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} -\left(R + r\cos u\right)\cos v \\ -\left(R + r\cos u\right)\sin v \\ 0 \end{pmatrix}$

$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \begin{pmatrix} \cos u\cos v \\ \cos u\sin v \\ \sin u \end{pmatrix}$ から，$\begin{cases} L = \sigma_{uu} \cdot \mathbf{n} = -r \\ M = \sigma_{uv} \cdot \mathbf{n} = 0 \\ N = \sigma_{vv} \cdot \mathbf{n} = -\left(R + r\cos u\right)\cos u \end{cases}$

$\mathbf{II_q}(\xi) = L\xi_1^2 + 2M\xi_1\xi_2 + N\xi_2^2 = -r\xi_1^2 - \left(R + r\cos u\right)\cos u\xi_2^2$

$$\lambda(\xi) = \frac{-r\xi_1^2 - \left(R + r\cos u\right)\cos u\xi_2^2}{r^2\xi_1^2 + \left(R + r\cos u\right)^2\xi_2^2} \tag{225}$$

ここで，$\beta : [0, 2\pi] \to \mathbb{R}$ を $\beta(\theta) := \lambda(\cos\theta, \sin\theta)$ で定めると，β が連続より，β は最大値 M，最小値 m が存在．$\forall\xi \in T_{\mathbf{q}}D = \mathbb{R}^2 \setminus \{0\}$ をとると，$\exists\begin{pmatrix} a \\ b \end{pmatrix} \in A$ と $S \in (0, \infty)$ で $\xi = S\begin{pmatrix} a \\ b \end{pmatrix}$ となる．このとき，$\lambda(\xi) = \lambda(sa, sb) = \lambda(a, b)$ より，$m \leq \lambda(\xi) \leq M$．したがって，$\lambda$ は $T_{\mathbf{q}}D \setminus \{0\}$ 上最大値と最小値を持つ．また

$$\lambda(\xi)\left(r^2\xi_1^2 + \left(R + r\cos u\right)^2\xi_2^2\right) - \left(r\xi_1^2 + \left(R + r\cos u\right)\cos u\xi_2^2\right) = 0 \tag{226}$$

$\xi = \begin{pmatrix} a \\ b \end{pmatrix}$ のとき，λ の最大値または最小値を取るとすると，$\dfrac{\partial\lambda}{\partial\xi_1}(a,b) = \dfrac{\partial\lambda}{\partial\xi_2}(a,b) = 0$

$\xi = \begin{pmatrix} a \\ b \end{pmatrix}$ で代入して微分すると $\begin{cases} \left(r^2\lambda(a,b) - r\right)a = 0 \\ \left((R + r\cos u)^2\lambda(a,b) - (R + r\cos u)\cos u\right)b = 0 \end{cases}$ よ

り，$\begin{pmatrix} a \\ b \end{pmatrix} \neq 0$ より，$\lambda(a,b) = \dfrac{1}{r}$ または $\lambda(a,b) = \dfrac{\cos u}{R + r\cos u} = \dfrac{1}{r}\left(1 - \dfrac{R}{R + r\cos u}\right)$

6 §7

6.1 E7.1

(1)

$$\sigma = \begin{pmatrix} (R + r\cos u)\cos v \\ (R + r\cos u)\sin v \\ r\sin u \end{pmatrix}, \sigma_u = \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix}, \sigma_v = \begin{pmatrix} -(R + r\cos u)\sin v \\ (R + r\cos u)\cos v \\ 0 \end{pmatrix}$$

$$\mathbf{e}_1 = \frac{\sigma_u}{\|\sigma_u\|} = \begin{pmatrix} -\sin u\cos v \\ -\sin u\sin v \\ \cos u \end{pmatrix} \tag{227}$$

$$\mathbf{e}_2 = \frac{\sigma_v - (\sigma_v \cdot \mathbf{e}_1)\,\mathbf{e}_1}{\|\sigma_v - (\sigma_v \cdot \mathbf{e}_1)\,\mathbf{e}_1\|} \tag{228}$$

ここで

$$\begin{aligned} &\sigma_v - (\sigma_v \cdot \mathbf{e}_1)\,\mathbf{e}_1 && (229)\\ &= \begin{pmatrix} -(R + r\cos u)\sin v \\ (R + r\cos u)\cos v \\ 0 \end{pmatrix} && (230)\\ &- \begin{pmatrix} -(R + r\cos u)\sin v \\ (R + r\cos u)\cos v \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin u\cos v \\ -\sin u\sin v \\ \cos u \end{pmatrix} \begin{pmatrix} -\sin u\cos v \\ -\sin u\sin v \\ \cos u \end{pmatrix} && (231)\\ &= \begin{pmatrix} -(R + r\cos u)\sin v \\ (R + r\cos u)\cos v \\ 0 \end{pmatrix} && (232)\end{aligned}$$

$$\|\sigma_v - (\sigma_v \cdot \mathbf{e}_1)\,\mathbf{e}_1\| = R + r\cos u \tag{233}$$

よって

$$\mathbf{e}_2 = \frac{1}{R + r\cos u}\begin{pmatrix} -(R + r\cos u)\sin v \\ (R + r\cos u)\cos v \\ 0 \end{pmatrix} \tag{234}$$

$$\sigma_u = r\mathbf{e}_1 + 0\mathbf{e}_2 \tag{235}$$

$$\sigma_v = 0\mathbf{e}_1 + 1\mathbf{e}_2 \tag{236}$$

よって

$$A = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} \tag{237}$$

$$= \begin{pmatrix} r & 0 \\ 0 & R + r\cos u \end{pmatrix} \tag{238}$$

(2)

$$\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} = A\begin{pmatrix} du \\ dv \end{pmatrix} \tag{239}$$

$$= \begin{pmatrix} r\mathrm{d}u \\ (R + r\cos u)\,\mathrm{d}v \end{pmatrix} \tag{240}$$

$$\begin{aligned} \mathrm{d}\theta^1 &= \mathrm{d}\left(r\mathrm{d}u\right) = r\mathrm{d}\left(\mathrm{d}u\right) && (241)\\ \mathrm{d}\theta^2 &= \mathrm{d}\left(\left(R + r\cos u\right)\mathrm{d}v\right) && (242)\\ &= \mathrm{d}\left(R + r\cos u\right)\wedge\mathrm{d}v + \left(R + r\cos u\right)\mathrm{d}\left(\mathrm{d}v\right) && (243)\\ &= -r\sin u\mathrm{d}u\wedge\mathrm{d}v && (244) \end{aligned}$$

(3)

(2) より $\begin{cases} -\omega\wedge\theta^2 = 0 \\ \omega\wedge\theta^1 = -r\sin u\mathrm{d}u\wedge\mathrm{d}v \end{cases}$

$\omega := f\mathrm{d}u + g\mathrm{d}v$ とすると $\begin{cases} \left(-f\mathrm{d}u - g\mathrm{d}v\right)\wedge\left(R + r\cos u\right)\mathrm{d}v = 0 \\ \left(f\mathrm{d}u + g\mathrm{d}v\right)\wedge r\mathrm{d}u = -r\sin u\mathrm{d}u\wedge\mathrm{d}v \end{cases}$

$\Longrightarrow \begin{cases} f\left(R + r\cos u\right)\mathrm{d}u\wedge\mathrm{d}v = 0 \\ -gr\mathrm{d}u\wedge\mathrm{d}v = -r\sin u\mathrm{d}u\wedge\mathrm{d}v \end{cases} \quad \Longrightarrow \begin{cases} f = 0 \\ g = \sin u \end{cases}$

よって，$\omega = \sin u\mathrm{d}v$ である

(4)

$\sigma_u = \begin{pmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{pmatrix}, \sigma_v = \begin{pmatrix} -\left(R + r\cos u\right)\sin v \\ \left(R + r\cos u\right)\cos v \\ 0 \end{pmatrix}, \begin{cases} E = \sigma_u\cdot\sigma_u = r^2 \\ F = \sigma_u\cdot\sigma_v = 0 \\ G = \sigma_v\cdot\sigma_v = \left(R + r\cos u\right)^2 \end{cases}$

$\sigma_u\times\sigma_v = \begin{pmatrix} -r\left(R + r\cos u\right)\cos u\cos v \\ -r\left(R + r\cos u\right)\cos u\sin v \\ -r\left(R + r\cos u\right)\sin u \end{pmatrix}, \|\sigma_u\times\sigma_v\| = r\left(R + r\cos u\right)$

$\mathbf{n} = \dfrac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|} = \begin{pmatrix} -\cos u\cos v \\ -\cos u\sin v \\ -\sin u \end{pmatrix}$

$\sigma_{uu} = \begin{pmatrix} -r\cos u\cos v \\ -r\cos u\sin v \\ -r\sin u \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} r\sin u\sin v \\ -r\sin u\cos v \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} -\left(R + r\cos u\right)\cos v \\ -\left(R + r\cos u\right)\sin v \\ 0 \end{pmatrix}$

$\Longrightarrow \begin{cases} L = \sigma_{uu}\cdot\mathbf{n} = r \\ M = \sigma_{uv}\cdot\mathbf{n} = 0 \\ N = \sigma_{vv}\cdot\mathbf{n} = \left(R + r\cos u\right)\cos u \end{cases} \quad \Longrightarrow K\circ\sigma = \dfrac{LN - M^2}{EG - F^2} = \dfrac{\cos u}{r\left(R + r\cos u\right)}$

$$\begin{aligned} \left(K\circ\sigma\right)\theta^1\wedge\theta^2 &= \frac{\cos u}{r\left(R + r\cos u\right)}\theta^1\wedge\theta^2 && (245)\\ &= \frac{\cos u}{r\left(R + r\cos u\right)} r\left(R + r\cos u\right)\mathrm{d}u\wedge\mathrm{d}v && (246)\\ &= \cos u\mathrm{d}u\wedge\mathrm{d}v && (247)\\ \mathrm{d}\omega &= \mathrm{d}\left(\sin u\mathrm{d}v\right) && (248)\\ &= \cos u\mathrm{d}u\wedge\mathrm{d}v && (249) \end{aligned}$$

よって，$\mathrm{d}\omega = \left(K\circ\sigma\right)\theta^1\wedge\theta^2$

6.2 P7.1

(1)

$$\sigma = \begin{pmatrix} u\cos v \\ u\sin v \\ kv \end{pmatrix}, \sigma_u = \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} -u\sin v \\ u\cos v \\ k \end{pmatrix}$$

$$\mathbf{e}_1 = \frac{\sigma_u}{\|\sigma_u\|} \tag{250}$$

$$= \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix} \tag{251}$$

$$\mathbf{e}_2 = \frac{\sigma_v - (\sigma_v \cdot \mathbf{e}_1)\,\mathbf{e}_1}{\|\sigma_v - (\sigma_v \cdot \mathbf{e}_1)\,\mathbf{e}_1\|} \tag{252}$$

ここで

$$\sigma_v - (\sigma_v \cdot \mathbf{e}_1)\,\mathbf{e}_1 = \begin{pmatrix} -u\sin v \\ u\cos v \\ k \end{pmatrix} - \begin{pmatrix} -u\sin v \\ u\cos v \\ k \end{pmatrix} \cdot \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix} \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix} \tag{253}$$

$$= \begin{pmatrix} -u\sin v \\ u\cos v \\ k \end{pmatrix} \tag{254}$$

から，$\mathbf{e}_2 = \frac{1}{\sqrt{u^2+k^2}} \begin{pmatrix} -u\sin v \\ u\cos v \\ k \end{pmatrix}$

$$\sigma_u = 1\mathbf{e}_1 + 0\mathbf{e}_2 \tag{255}$$

$$\sigma_v = 0\mathbf{e}_1 + \sqrt{u^2+k^2}\mathbf{e}_2 \tag{256}$$

よって

$$A = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} \tag{257}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{u^2+k^2} \end{pmatrix} \tag{258}$$

(2)

$$\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} = A \begin{pmatrix} du \\ dv \end{pmatrix} \tag{259}$$

$$= \begin{pmatrix} \mathrm{d}u \\ \sqrt{u^2+k^2}\mathrm{d}v \end{pmatrix} \tag{260}$$

から

$$\mathrm{d}\theta^1 = \mathrm{d}\,(\mathrm{d}u) = 0 \tag{261}$$

$$\mathrm{d}\theta^2 = \mathrm{d}\left(\sqrt{u^2+k^2}\mathrm{d}v\right) \tag{262}$$

$$= \frac{u}{\sqrt{u^2+k^2}}\mathrm{d}u \wedge \mathrm{d}v \tag{263}$$

(3)

$\omega = f\mathrm{d}u + g\mathrm{d}v$ とすると

$$-\omega \wedge \theta^2 = -\left(f\mathrm{d}u + g\mathrm{d}v\right) \wedge \sqrt{u^2+k^2}\mathrm{d}v = -f\sqrt{u^2+k^2}\mathrm{d}u \wedge \mathrm{d}v \tag{264}$$

$$\omega \wedge \theta^1 = \left(f\mathrm{d}u + g\mathrm{d}v\right) \wedge \mathrm{d}u = -g\mathrm{d}v \wedge \mathrm{d}u \tag{265}$$

$$\Longrightarrow \begin{cases} f = 0 \\ g = -\dfrac{u}{\sqrt{u^2+k^2}} \end{cases} \quad \Longrightarrow \omega = -\frac{u}{\sqrt{u^2+k^2}}\mathrm{d}v$$

(4)

$$\sigma = \begin{pmatrix} u\cos v \\ u\sin v \\ kv \end{pmatrix}, \sigma_u = \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix}, \sigma_v = \begin{pmatrix} -u\sin v \\ u\cos v \\ k \end{pmatrix} \Longrightarrow \begin{cases} E = \sigma_u \cdot \sigma_u = 1 \\ F = \sigma_u \cdot \sigma_v = 0 \\ G = \sigma_v \cdot \sigma_v = u^2 + k^2 \end{cases}$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} k\sin v \\ -k\cos v \\ u \end{pmatrix}, \|\sigma_u \times \sigma_v\| = \sqrt{u^2+k^2}, \mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{u^2+k^2}}\begin{pmatrix} k\sin v \\ -k\cos v \\ u \end{pmatrix}$$

$$\sigma_{uu} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma_{uv} = \begin{pmatrix} -\sin v \\ \cos v \\ 0 \end{pmatrix}, \sigma_{vv} = \begin{pmatrix} -u\cos v \\ -u\sin v \\ 0 \end{pmatrix} \Longrightarrow \begin{cases} L = \sigma_{uu} \cdot \mathbf{n} = 0 \\ M = \sigma_{uv} \cdot \mathbf{n} = -\dfrac{k}{\sqrt{u^2+k^2}} \\ N = \sigma_{vv} \cdot \mathbf{n} = 0 \end{cases}$$

$K \circ \sigma = \dfrac{LN - M^2}{EG - F^2} = \dfrac{k^2}{\left(u^2+k^2\right)^2}$ から，$\left(K \circ \sigma\right)\theta^1 \wedge \theta^2 = \dfrac{k^2}{u^2+k^2}\mathrm{d}u \wedge \mathrm{d}v = \mathrm{d}\omega$

7 §8

7.1 E8.1

(1)

$g_q(\xi,\eta) = \left(1+u^2\right)\xi_1\eta_1 + uv\left(\xi_1\eta_2+\xi_2\eta_1\right) + \left(1+v^2\right)\xi_2\eta_2 \Longrightarrow g_q(\xi,\eta) = {}^t\xi\begin{pmatrix} 1+u^2 & uv \\ uv & 1+v^2 \end{pmatrix}\eta$
$1+u^2 \geq 1 > 0$ で，$\left(1+u^2\right)\left(1+v^2\right) - u^2v^2 = u^2+v^2+1 > 0$ が成り立つから，$g_q(\xi,\eta)$ は正定値である．また，$1+u^2, uv, 1+v^2$ は C^∞ であるので，$g_q(\xi,\eta)$ は Riemann 計量となる

(2)

$$\epsilon_1(u,v) = \frac{\partial_1}{\|\partial_1\|_q} = \frac{1}{\sqrt{1+u^2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{266}$$

$$\begin{aligned}
(\partial_2,\epsilon_1)_q = g_q(\partial_1,\epsilon_1) &= g_q\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{1+u^2}} \\ 0 \end{pmatrix}\right) && (267)\\
&= \left(1+u^2\right)\cdot 0 + uv\cdot\frac{1}{\sqrt{1+u^2}} + \left(1+v^2\right)\cdot 0 && (268)\\
&= \frac{uv}{\sqrt{1+u^2}} && (269)
\end{aligned}$$

$$\epsilon_2(u,v) = \frac{\partial_2 - (\partial_2,\epsilon_1)_q\epsilon_1}{\left\|\partial_2 - (\partial_2,\epsilon_1)_q\epsilon_1\right\|_q} \tag{270}$$

で

$$\begin{aligned}
\partial_2 - (\partial_2,\epsilon_1)_q\epsilon_1 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{uv}{\sqrt{1+u^2}}\cdot\frac{1}{\sqrt{1+u^2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} && (271)\\
&= \begin{pmatrix} -\frac{uv}{1+u^2} \\ 1 \end{pmatrix} && (272)\\
\left\|\partial_2 - (\partial_2,\epsilon_1)_q\epsilon_1\right\|_q^2 &= g_q(\epsilon_2,\epsilon_2) && (273)\\
&= \left(1+u^2\right)\cdot\frac{u^2v^2}{\left(1+u^2\right)^2} + uv\cdot\left(-\frac{2uv}{1+u^2}\right) + \left(1+v^2\right)\cdot 1 && (274)\\
&= \frac{1+u^2+v^2}{1+u^2} && (275)
\end{aligned}$$

から

$$\epsilon_2(u,v) = \sqrt{\frac{1+u^2}{1+u^2+v^2}}\begin{pmatrix} -\frac{uv}{1+u^2} \\ 1 \end{pmatrix} \tag{276}$$

よって

$$A = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} = (\epsilon_1, \epsilon_2)^{-1} \tag{277}$$

$$= \begin{pmatrix} \dfrac{1}{\sqrt{1+u^2}} & -\dfrac{uv}{1+u^2}\sqrt{\dfrac{1+u^2}{1+u^2+v^2}} \\ 0 & \sqrt{\dfrac{1+u^2}{1+u^2+v^2}} \end{pmatrix}^{-1} \tag{278}$$

$$= \begin{pmatrix} \sqrt{1+u^2} & \dfrac{uv}{\sqrt{1+u^2}} \\ 0 & \sqrt{\dfrac{1+u^2+v^2}{1+u^2}} \end{pmatrix} \tag{279}$$

(3)

$$\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} = A \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix} \tag{280}$$

$$= \begin{pmatrix} \sqrt{1+u^2} & \dfrac{uv}{\sqrt{1+u^2}} \\ 0 & \sqrt{\dfrac{1+u^2+v^2}{1+u^2}} \end{pmatrix} \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix} \tag{281}$$

$$= \begin{pmatrix} \sqrt{1+u^2}\mathrm{d}u + \dfrac{uv}{\sqrt{1+u^2}}\mathrm{d}v \\ \sqrt{\dfrac{1+u^2+v^2}{1+u^2}}\mathrm{d}v \end{pmatrix} \tag{282}$$

$$\begin{pmatrix} \mathrm{d}\theta^1 \\ \mathrm{d}\theta^2 \end{pmatrix} = \begin{pmatrix} \dfrac{v}{(1+u^2)^{\frac{3}{2}}} \\ -\dfrac{uv^2}{(1+u^2)^2}\sqrt{\dfrac{1+u^2}{1+u^2+v^2}} \end{pmatrix} \mathrm{d}u \wedge \mathrm{d}v \tag{283}$$

(4)

$\omega = f\mathrm{d}u + g\mathrm{d}v$ とすると

$$-\omega \wedge \theta^2 = -(f\mathrm{d}u + g\mathrm{d}v) \wedge \left(\sqrt{\frac{1+u^2+v^2}{1+u^2}}\mathrm{d}v\right) \tag{284}$$

$$= -f\sqrt{\frac{1+u^2+v^2}{1+u^2}}\mathrm{d}u \wedge \mathrm{d}v = -\mathrm{d}\theta^1 = -\frac{v}{(1+u^2)^{\frac{3}{2}}}\mathrm{d}u \wedge \mathrm{d}v \tag{285}$$

$$\omega \wedge \theta^1 = (f\mathrm{d}u + g\mathrm{d}v) \wedge \left(\sqrt{1+u^2}\mathrm{d}u + \frac{uv}{\sqrt{1+u^2}}\mathrm{d}v/\right) \tag{286}$$

$$= \left(\frac{fuv}{\sqrt{1+u^2}} - g\sqrt{1+u^2}\right)\mathrm{d}u \wedge \mathrm{d}v = -\mathrm{d}\theta^2 = \frac{uv^2}{(1+u^2)^2}\sqrt{\frac{1+u^2}{1+u^2+v^2}}\mathrm{d}u \wedge \mathrm{d}v \tag{287}$$

$\begin{cases} f\sqrt{\dfrac{1+u^2+v^2}{1+u^2}} = \dfrac{v}{(1+u^2)^{\frac{3}{2}}} \\ \dfrac{fuv}{\sqrt{1+u^2}} - g\sqrt{1+u^2} = \dfrac{uv^2}{(1+u^2)^2}\sqrt{\dfrac{1+u^2}{1+u^2+v^2}} \end{cases} \implies \begin{cases} f = \dfrac{v}{(1+u^2)\sqrt{1+u^2+v^2}} \\ g = 0 \end{cases}$ よって，$\omega = \dfrac{v}{(1+u^2)\sqrt{1+u^2+v^2}}\mathrm{d}u$

(5)

$\mathrm{d}\omega = \dfrac{1}{(1+u^2+v^2)^{\frac{3}{2}}}$ で

$$\theta^1 \wedge \theta^2 = \left(\sqrt{1+u^2}\mathrm{d}u + \frac{uv}{\sqrt{1+u^2}}\mathrm{d}v\right) \wedge \left(\sqrt{\frac{1+u^2+v^2}{1+u^2}}\mathrm{d}v\right) \tag{288}$$

$$= \sqrt{1+u^2+v^2}\mathrm{d}u \wedge \mathrm{d}v \tag{289}$$

から

$$K = \frac{\mathrm{d}\omega}{\theta^1 \wedge \theta^2} \tag{290}$$

$$= \frac{1}{(1+u^2+v^2)^2} \tag{291}$$

7.2 P8.1

(1)

$g_q(\xi,\eta) = \dfrac{4}{(1-u^2-v^2)^2}\xi_1\eta_1 + \dfrac{4}{(1-u^2-v^2)^2}\xi_2\eta_2$

$\Longrightarrow g_q(\xi,\eta) = {}^t\xi \begin{pmatrix} \dfrac{4}{(1-u^2-v^2)^2} & 0 \\ 0 & \dfrac{4}{(1-u^2-v^2)^2} \end{pmatrix} \eta$ から

$\left(\dfrac{4}{(1-u^2-v^2)^2}\right)^2 > 0$ で, $\dfrac{4}{(1-u^2-v^2)^2} > 0$ なので, $g_q(\xi,\eta)$ は正定値である. よって, $g_q(\xi,\eta)$ は Riemann 計量になる

(2)

$\epsilon_1 = \dfrac{\partial_1}{\|\partial_1\|_q} = \dfrac{|1-u^2-v^2|}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \dfrac{1-u^2-v^2}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, (u^2+v^2<1)$

$$\langle \partial_2, \epsilon_1 \rangle_q = g_q\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \dfrac{1-u^2-v^2}{2} \\ 0 \end{pmatrix}\right) = 0 \tag{292}$$

$$\partial_2 - (\partial_2, \epsilon_1)_q \epsilon_1 = \partial_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{293}$$

$$\epsilon_2 = \frac{\partial_2 - (\partial_2, \epsilon_1)_q \epsilon_1}{\left\|\partial_2 - (\partial_2, \epsilon_1)_q \epsilon_1\right\|_q} = \frac{1-u^2-v^2}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{294}$$

よって

$$A = \begin{pmatrix} \dfrac{1-u^2-v^2}{2} & 0 \\ 0 & \dfrac{1-u^2-v^2}{2} \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{2}{1-u^2-v^2} & 0 \\ 0 & \dfrac{2}{1-u^2-v^2} \end{pmatrix} \tag{295}$$

(3)

$$\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} = A \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix} \tag{296}$$

$$= \begin{pmatrix} \dfrac{2}{1-u^2-v^2} & 0 \\ 0 & \dfrac{2}{1-u^2-v^2} \end{pmatrix} \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix} \tag{297}$$

$$= \begin{pmatrix} \dfrac{2}{1-u^2-v^2}\mathrm{d}u \\ \dfrac{2}{1-u^2-v^2}\mathrm{d}v \end{pmatrix} \tag{298}$$

$$\begin{pmatrix} \mathrm{d}\theta^1 \\ \mathrm{d}\theta^2 \end{pmatrix} = \begin{pmatrix} -\dfrac{4v}{(1-u^2-v^2)^2} \\ \dfrac{4u}{(1-u^2-v^2)^2} \end{pmatrix} \mathrm{d}u \wedge \mathrm{d}v \tag{299}$$

(3)

$\omega = f\mathrm{d}u + g\mathrm{d}v$ とする

$$-\omega \wedge \theta^2 = -(f\mathrm{d}u + g\mathrm{d}v) \wedge \left(\frac{2}{1-u^2-v^2}\mathrm{d}v\right) \tag{300}$$

$$= -\frac{2f}{1-u^2-v^2}\mathrm{d}u \wedge \mathrm{d}v = \mathrm{d}\theta^1 = -\frac{4v}{(1-u^2-v^2)^2}\mathrm{d}u \wedge \mathrm{d}v \tag{301}$$

$$\omega \wedge \theta^1 = (f\mathrm{d}u + g\mathrm{d}v) \wedge \left(\frac{2}{1-u^2-v^2}\mathrm{d}u\right) \tag{302}$$

$$= -\frac{2g}{1-u^2-v^2}\mathrm{d}u \wedge \mathrm{d}v = \mathrm{d}\theta^2 = \frac{4u}{(1-u^2-v^2)^2}\mathrm{d}u \wedge \mathrm{d}v \tag{303}$$

$$\begin{cases} \dfrac{2f}{1-u^2-v^2} = \dfrac{4v}{(1-u^2-v^2)^2} \\ -\dfrac{2g}{1-u^2-v^2} = \dfrac{4u}{(1-u^2-v^2)^2} \end{cases} \Longrightarrow \begin{cases} f = \dfrac{2v}{(1-u^2-v^2)} \\ g = -\dfrac{2u}{(1-u^2-v^2)} \end{cases}$$
$$\Longrightarrow \omega = \frac{2v}{(1-u^2-v^2)}\mathrm{d}u - \frac{2u}{(1-u^2-v^2)}\mathrm{d}v$$

(5)

$\mathrm{d}\omega = -\dfrac{4}{(1-u^2-v^2)^2}\mathrm{d}u \wedge \mathrm{d}v$

$\theta^1 \wedge \theta^2 = \dfrac{4}{(1-u^2-v^2)^2}\mathrm{d}u \wedge \mathrm{d}v$

$$K = \frac{\mathrm{d}\omega}{\theta^1 \wedge \theta^2} = -1 \tag{304}$$

8 §9

8.1 E9.1

(1)

$\mathbf{a}, \mathbf{b} \in S^2(r)$ より $\|\mathbf{a}\| = \|\mathbf{b}\| = r$．よって，$\mathbf{a} \cdot \mathbf{b} = 0$ より，$\forall t \in \mathbb{R}$

$$\begin{aligned} \|\gamma(t)\| &= \sqrt{\gamma(t) \cdot \gamma(t)} && (305) \\ &= \sqrt{\|\mathbf{a}\|^2 \cos^2 t + \|\mathbf{b}\|^2 \sin^2 t + 2(\mathbf{a} \cdot \mathbf{b}) \sin t \cos t} && (306) \\ &= \sqrt{r^2 \cos^2 t + r^2 \sin^2 t} = r && (307) \end{aligned}$$

となるので，$\gamma(t) \in S^2(r)$．よって，γ は $S^2(r)$ 上の曲線である

(2)

$\forall t \in \mathbb{R}, \ddot{\gamma}(t) = -\mathbf{a}\cos t - \mathbf{b}\sin t = -\gamma(t), \mathbf{n}(\mathbf{p}) = \dfrac{1}{r}\mathbf{p}, \mathbf{n}(\gamma(t)) = \dfrac{1}{r}\gamma(t)$
C^∞ 関数 $f : \mathbf{R}^3 \to \mathbf{R}$ を $f(x, y, z) = x^2 + y^2 + z^2 - r^2$ で定めると，$\forall \mathbf{p} \in S^2(r), f$ は $S^2(r)$ の局所方程式．このとき，$\mathbf{p} \in S^2(r)$ に対し

$$(\nabla f)(\mathbf{p}) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \tag{308}$$

より，$(\nabla f)(\gamma(t)) = 2\gamma(t)$．よって，$T_{\gamma(t)}S^2(r) = \left\{\mathbf{v} \in \mathbb{R}^3 \middle| \gamma(t) \cdot \mathbf{v} = 0\right\}$．従って，$\mathbf{v} \in T_{\gamma(t)}S^2(r)$ に対して，$\ddot{\gamma}(t) \cdot \mathbf{v} = -(\gamma(t) \cdot \mathbf{v}) = 0$ から，γ は測地線

8.2 E9.2

S の σ に関する第一基本量は

$$E(u^1, u^2) = 1, \quad F(u^1, u^2) = 0, \quad G(u^1, u^2) = (u^1)^2 + k^2 \tag{309}$$

従って

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & (u^1)^2 + k^2 \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{(u^1)^2 + k^2} \end{pmatrix} \tag{310}$$

である．$\forall i, j, k \in \{1, 2\}$ に対して，$\dfrac{\partial g_{ij}}{\partial u^k} = \begin{cases} 2u^1 & (i,j,k) = (2,2,1) \\ 0 & \text{その他} \end{cases}$．ゆえに，$\forall i, j, k \in \{1, 2\}$ に対して

$$\frac{\partial g_{jl}}{\partial u^k} + \frac{\partial g_{kl}}{\partial u^j} - \frac{\partial g_{kj}}{\partial u^l} = \begin{cases} 2u^1 & (j,k,l) = (1,2,2), (2,1,2) \\ -2u^1 & (j,k,l) = (2,2,1) \\ 0 & \text{その他} \end{cases} \tag{311}$$

よって，$\forall i,j,k \in \{1,2\}$ に対して

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^{2} g^{li}\left(\frac{\partial g_{jl}}{\partial u^k} + \frac{\partial g_{kl}}{\partial u^j} - \frac{\partial g_{kj}}{\partial u^l}\right) \tag{312}$$

$$= \frac{1}{2} g^{ii}\left(\frac{\partial g_{ji}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{kj}}{\partial u^i}\right) \tag{313}$$

$$= \begin{cases} \frac{u^1}{\left(u^1\right)^2 + k^2} & (i,j,k) = (2,1,2), (2,2,1) \\ -u^1 & (i,j,k) = (1,2,2) \\ 0 & \text{その他} \end{cases} \tag{314}$$

8.3 E9.3

S の σ に関する第一基本量は

$$E\left(u^1,u^2\right) = a^2\left(u^1\right)^2 + 1, \quad F\left(u^1,u^2\right) = abu^1u^2, \quad G\left(u^1,u^2\right) = b^2\left(u^2\right)^2 + 1 \tag{315}$$

従って

$$\left(g_{ij}\right) = \begin{pmatrix} a^2\left(u^1\right)^2 + 1 & abu^1u^2 \\ abu^1u^2 & b^2\left(u^2\right)^2 + 1 \end{pmatrix} \tag{316}$$

$$\left(g^{ij}\right) = \frac{1}{a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2 + 1}\begin{pmatrix} b^2\left(u^2\right)^2 + 1 & -abu^1u^2 \\ -abu^1u^2 & a^2\left(u^1\right)^2 + 1 \end{pmatrix} \tag{317}$$

$$= \begin{pmatrix} 1 - \frac{a^2\left(u^1\right)^2}{a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2 + 1} & -\frac{abu^1u^2}{a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2 + 1} \\ -\frac{abu^1u^2}{a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2 + 1} & 1 - \frac{b^2\left(u^2\right)^2}{a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2 + 1} \end{pmatrix} \tag{318}$$

となる．$c_1 = a, c_2 = b$ とおくと $\begin{cases} g_{ij} = \delta_{ij} + c_ic_ju^iu^j \\ g^{ij} = \delta_{ij} - \frac{c_ic_ju^iu^j}{a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2 + 1} \end{cases}$ $(i,j \in \{1,2\})$ である

$\forall i,j,k \in \{1,2\}$ に対して，$\frac{\partial g_{ij}}{\partial u^k} = c_ic_j\left(\delta_{ik}u^j + \delta_{jk}u^i\right)$．ゆえに，$\forall i,j,k \in \{1,2\}$ に対して

$$\frac{\partial g_{jl}}{\partial u^k} + \frac{\partial g_{kl}}{\partial u^j} - \frac{\partial g_{kj}}{\partial u^l} = c_jc_l\left(\delta_{jk}u^l + \delta_{lk}u^j\right) + c_kc_l\left(\delta_{kj}u^l + \delta_{lj}u^k\right) - c_kc_j\left(\delta_{kl}u^j + \delta_{jl}u^k\right) \tag{319}$$

$$= c_l\left(c_j + c_k\right)\delta_{jk}u^l + c_k\left(c_l - c_j\right)\delta_{jl}u^k + c_j\left(c_l - c_k\right)\delta_{kl}u^j \tag{320}$$

$$= c_l\left(c_j + c_k\right)\delta_{jk}u^l \tag{321}$$

よって，$\forall i,j,k \in \{1,2\}$ に対して

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^{2} g^{li}\left(\frac{\partial g_{jl}}{\partial u^k} + \frac{\partial g_{kl}}{\partial u^j} - \frac{\partial g_{kj}}{\partial u^l}\right) \tag{322}$$

$$= \frac{1}{2}\left(c_j + c_k\right)\delta_{jk}\sum_{l=1}^{2}\left(\delta_{li} - \frac{c_lc_iu^lu^i}{a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2 + 1}\right)c_lu^l \tag{323}$$

$$= \frac{1}{2}\left(c_j + c_k\right)\delta_{jk}\left(c_iu^i - c_iu^i \cdot \frac{a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2}{a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2 + 1}\right) \tag{324}$$

$$= \frac{c_i\left(c_j + c_k\right)\delta_{jk}u^i}{2\left(a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2 + 1\right)} \tag{325}$$

ゆえに

$$\Gamma^i_{jk} = \begin{cases} \dfrac{a^2u^1}{a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2 + 1} & (i,j,k) = (1,1,1) \\ \dfrac{abu^1}{a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2 + 1} & (i,j,k) = (1,2,2) \\ \dfrac{abu^2}{a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2 + 1} & (i,j,k) = (2,1,1) \\ \dfrac{b^2u^2}{a^2\left(u^1\right)^2 + b^2\left(u^2\right)^2 + 1} & (i,j,k) = (2,2,2) \\ 0 & \text{その他} \end{cases} \tag{326}$$

8.4 P9.1

(1)

$x^2 + y^2 = (r\cos t)^2 + (r\sin t)^2 = r^2 \Longrightarrow \gamma(t) \in S$

(2)

$\forall t \in \mathbb{R}, \ddot{\gamma}(t) = \begin{pmatrix} -r\cos t \\ -r\sin t \\ 0 \end{pmatrix}, f(x,y,z) = x^2 + y^2 - r^2$ とおくと，$\nabla f = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}$ で，γ を代入すると

$$(\nabla f)(\gamma(t)) = \begin{pmatrix} 2r\cos t \\ 2r\sin t \\ 0 \end{pmatrix} \tag{327}$$

から, $\ddot{\gamma}(t) = -\dfrac{1}{2}(\nabla f)(\gamma(t))$ で, $T_{\gamma(t)}S = \left\{\mathbf{v} \in \mathbb{R}^3 \middle| (\nabla f)(\gamma(t)) \cdot \mathbf{v} = 0\right\} = \left\{\mathbf{v} \in \mathbb{R}^3 \middle| \ddot{\gamma}(t) \cdot \mathbf{v} = 0\right\}$ から，$\gamma(t) \perp T_{\gamma(t)}S$ で，測地線になる

8.5 P9.2

S の σ に関する第一基本量は

$$E\left(u^1, u^2\right) = r^2, \quad F\left(u^1, u^2\right) = 0, \quad G\left(u^1, u^2\right) = (R + r\cos u)^2 \tag{328}$$

従って

$$\left(g_{ij}\right) = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r\cos u)^2 \end{pmatrix} \tag{329}$$

$$\left(g^{ij}\right) = \begin{pmatrix} \dfrac{1}{r^2} & 0 \\ 0 & \dfrac{1}{(R + r\cos u)^2} \end{pmatrix} \tag{330}$$

である．$\forall i,j,k\in\{1,2\}$ に対して，$\frac{\partial g_{ij}}{\partial u^k}=\begin{cases}-2r\left(R+r\cos u\right)\sin u & (i,j,k)=(2,2,1)\\ 0 & \text{その他}\end{cases}$

δ 関数を用いると

$$\left(g_{ij}\right)=r^2\delta_{i1}\delta_{j1}+\left(R+r\cos u^1\right)^2\delta_{i2}\delta_{j2} \tag{331}$$

$$\left(g^{ij}\right)=\frac{1}{r^2}\delta_{i1}\delta_{j1}+\frac{1}{\left(R+r\cos u^1\right)^2}\delta_{i2}\delta_{j2} \tag{332}$$

$$\frac{\partial g_{ij}}{\partial u^k}=-2r\left(R+r\cos u^1\right)\sin u^1\delta_{i2}\delta_{j2}\delta_{k1} \tag{333}$$

$$\frac{\partial g_{jl}}{\partial u^k}+\frac{\partial g_{kl}}{\partial u^j}-\frac{\partial g_{kj}}{\partial u^l}=-2r\left(R+r\cos u^1\right)\sin u^1\left(\delta_{j2}\delta_{l2}\delta_{k1}+\delta_{k2}\delta_{l2}\delta_{j1}-\delta_{k2}\delta_{j2}\delta_{l1}\right) \tag{334}$$

$$\Gamma^i_{jk}=\frac{1}{2}\sum_{l=1}^{2}g^{li}\left(\frac{\partial g_{jl}}{\partial u^k}+\frac{\partial g_{kl}}{\partial u^j}-\frac{\partial g_{kj}}{\partial u^l}\right) \tag{335}$$

$$=\frac{1}{2}\sum_{l=1}^{2}\left(\frac{1}{r^2}\delta_{l1}\delta_{i1}+\frac{1}{\left(R+r\cos u^1\right)^2}\delta_{l2}\delta_{i2}\right) \tag{336}$$

$$\cdot\left(\delta_{j2}\delta_{l2}\delta_{k1}+\delta_{k2}\delta_{l2}\delta_{j1}-\delta_{k2}\delta_{j2}\delta_{l1}\right)\cdot\left(-2r\left(R+r\cos u^1\right)\sin u^1\right) \tag{337}$$

$$=\frac{1}{2}\sum_{l=1}^{2}\left(-\frac{1}{r^2}\delta_{i1}\delta_{j2}\delta_{k2}\delta_{l1}+\frac{\delta_{i2}\delta_{l2}\left(\delta_{j1}\delta_{k2}+\delta_{j2}\delta_{k1}\right)}{\left(R+r\cos u^1\right)^2}\right)\left(-2r\left(R+r\cos u^1\right)\sin u^1\right) \tag{338}$$

$$\begin{cases}-\frac{1}{r^2}\delta_{i1}\delta_{j2}\delta_{k2}\cdot\left(-2r\left(R+r\cos u^1\right)\sin u^1\right) & l=1\\ \frac{1}{\left(R+r\cos u^1\right)^2}\delta_{i2}\left(\delta_{j1}\delta_{k2}+\delta_{j2}\delta_{k1}\right)\cdot\left(-2r\left(R+r\cos u^1\right)\sin u^1\right) & l=2\end{cases}\ \text{から}$$

$$\Gamma^i_{jk}=-r\left(R+r\cos u^1\right)\sin u^1\left(-\frac{1}{r^2}\delta_{i1}\delta_{j2}\delta_{k2}+\frac{1}{\left(R+r\cos u^1\right)^2}\delta_{i2}\left(\delta_{j1}\delta_{k2}+\delta_{j2}\delta_{k1}\right)\right) \tag{339}$$

$$=\begin{cases}\frac{1}{r}\left(R+r\cos u^1\right)\sin u^1 & (i,j,k)=(1,2,2)\\ -\frac{r\sin u^1}{R+r\cos u^1} & (i,j,k)=(2,1,2),(2,2,1)\\ 0 & \text{その他}\end{cases} \tag{340}$$

8.6 P9.3

S の σ に関する第一基本量は

$$E\left(u^1,u^2\right)=9u^4+1,\quad F\left(u^1,u^2\right)=0,\quad G\left(u^1,u^2\right)=1 \tag{341}$$

従って

$$\left(g_{ij}\right)=\begin{pmatrix}9\left(u^1\right)^4+1 & 0\\ 0 & 1\end{pmatrix}=\left(9u^4+1\right)\delta_{i1}\delta_{j1}+\delta_{i2}\delta_{j2} \tag{342}$$

$$\left(g^{ij}\right)=\begin{pmatrix}\frac{1}{9\left(u^1\right)^4+1} & 0\\ 0 & 1\end{pmatrix}=\frac{1}{9u^4+1}\delta_{i1}\delta_{j1}+\delta_{i2}\delta_{j2} \tag{343}$$

である．$\forall i, j, k \in \{1, 2\}$ に対して，$\dfrac{\partial g_{ij}}{\partial u^k} = \begin{cases} 36\left(u^1\right)^3 & (i, j, k) = (1, 1, 1) \\ 0 & \text{その他} \end{cases}$

$$\frac{\partial g_{jl}}{\partial u^k} + \frac{\partial g_{kl}}{\partial u^j} - \frac{\partial g_{kj}}{\partial u^l} = 36\left(u^1\right)^3 \left(\delta_{j1}\delta_{k1}\delta_{l1} + \delta_{j1}\delta_{k1}\delta_{l1} - \delta_{j1}\delta_{k1}\delta_{l1}\right) \tag{344}$$

$$= 36\left(u^1\right)^3 \delta_{j1}\delta_{k1}\delta_{l1} \tag{345}$$

よって，$\forall i, j, k \in \{1, 2\}$ に対して

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^{2} g^{li}\left(\frac{\partial g_{jl}}{\partial u^k} + \frac{\partial g_{kl}}{\partial u^j} - \frac{\partial g_{kj}}{\partial u^l}\right) \tag{346}$$

$$= \frac{1}{2}\sum_{l=1}^{2} 36\left(u^1\right)^3 \left(\frac{1}{9u^4+1}\delta_{l1}\delta_{i1} + \delta_{l2}\delta_{i2}\right)\delta_{j1}\delta_{k1}\delta_{l1} \tag{347}$$

$$= \frac{1}{2}\sum_{l=1}^{2} \frac{36u^3}{9u^4+1}\left(\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1} + \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1} - \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1}\right) \tag{348}$$

$\begin{cases} \dfrac{36u^3}{9u^4+1}\delta_{i1}\delta_{j1}\delta_{k1} & l = 1 \\ 0 & l = 2 \end{cases}$ から

$$\Gamma^i_{jk} = \frac{18u^2}{9u^4+1}\delta_{i1}\delta_{j1}\delta_{k1} \tag{349}$$

$$= \begin{cases} \dfrac{18\left(u^1\right)^3}{9\left(u^1\right)^4+1} & (i, j, k) = (1, 1, 1) \\ 0 & \text{その他} \end{cases} \tag{350}$$

9 §10

9.1 E10.1

(1)

$\forall t \in I$

$$X(t) \cdot \mathbf{n}(\gamma(t)) = \begin{pmatrix} h\cos t - t^2 \sin t \\ h \sin t + t^2 \cos t \\ -\sqrt{r^2 - h^2} \end{pmatrix} \cdot \frac{1}{r} \begin{pmatrix} \sqrt{r^2 - h^2}\cos t \\ \sqrt{r^2 - h^2}\sin t \\ h \end{pmatrix} \tag{351}$$

$$= \frac{1}{r}\left(h\sqrt{r^2 - h^2}\cos^2 t + h\sqrt{r^2 - h^2}\sin^2 t - h\sqrt{r^2 - h^2}\right) = 0 \tag{352}$$

より，$X(t) \in T_{\gamma(t)} S^2(r)$．故に，$X$ は γ に沿う接ベクトル場である．また

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \begin{pmatrix} -t^2 \cos t - (2t + h)\sin t \\ -t^2 \sin t + (2t + h)\cos t \\ 0 \end{pmatrix} \tag{353}$$

より，$\dfrac{\mathrm{d}X}{\mathrm{d}t}(t) \cdot \mathbf{n}(\gamma(t)) = -\dfrac{\sqrt{r^2 - h^2}}{r} t^2$．従って

$$\frac{\mathrm{D}X}{\mathrm{d}t}(t) = \left(\frac{\mathrm{d}X}{\mathrm{d}t}(t)\right)_{tan} \tag{354}$$

$$= \frac{\mathrm{d}X}{\mathrm{d}t}(t) - \left(\frac{\mathrm{d}X}{\mathrm{d}t}(t) \cdot \mathbf{n}(\gamma(t))\right)\mathbf{n}(\gamma(t)) \tag{355}$$

$$= \begin{pmatrix} -t^2 \cos t - (2t + h)\sin t \\ -t^2 \sin t + (2t + h)\cos t \\ 0 \end{pmatrix} + \frac{\sqrt{r^2 - h^2}}{r} t^2 \cdot \frac{1}{r} \begin{pmatrix} \sqrt{r^2 - h^2}\cos t \\ \sqrt{r^2 - h^2}\sin t \\ h \end{pmatrix} \tag{356}$$

$$= \begin{pmatrix} -\dfrac{h^2}{r^2} t^2 \cos t - (2t + h)\sin t \\ -\dfrac{h^2}{r^2} t^2 \sin t + (2t + h)\cos t \\ \dfrac{h\sqrt{r^2 - h^2}}{r^2} t^2 \end{pmatrix} \tag{357}$$

また，第三成分が常に 0 になるのは $h = 0$ のとき，第一成分，第二成分は $-2t\sin t, 2t\cos t$ なので，例えば $t = \pi$ のとき，両方が 0 ではない．すなわち，$\dfrac{\mathrm{D}X}{\mathrm{d}t}(t) \not\equiv 0$．従って，$X$ は γ に沿って平行ではない

(2)

$\forall t \in I$

$$\dot{\gamma}(t) \cdot \mathbf{n}(\gamma(t)) = \begin{pmatrix} -\sqrt{r^2 - h^2}\sin t \\ \sqrt{r^2 - h^2}\cos t \\ 0 \end{pmatrix} \cdot \frac{1}{r} \begin{pmatrix} \sqrt{r^2 - h^2}\cos t \\ \sqrt{r^2 - h^2}\sin t \\ h \end{pmatrix} = 0 \tag{358}$$

より,$\dot{\gamma}(t) \in T_{\gamma(t)}S^2(r)$.よって,$\dot{\gamma}$ は γ に沿う接ベクトル場である.また $\dfrac{\mathrm{d}\dot{\gamma}}{\mathrm{d}t}(t) = \begin{pmatrix} -\sqrt{r^2-h^2}\cos t \\ -\sqrt{r^2-h^2}\sin t \\ 0 \end{pmatrix}$

より,$\dfrac{\mathrm{d}\dot{\gamma}}{\mathrm{d}t}(t) \cdot \mathbf{n}(\gamma(t)) = -\dfrac{r^2-h^2}{r}$.従って

$$\frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t) = \frac{\mathrm{d}\dot{\gamma}}{\mathrm{d}t}(t) - \left(\frac{\mathrm{d}\dot{\gamma}}{\mathrm{d}t}(t) \cdot \mathbf{n}(\gamma(t))\right)\mathbf{n}(\gamma(t)) \tag{359}$$

$$= \begin{pmatrix} -\sqrt{r^2-h^2}\cos t \\ -\sqrt{r^2-h^2}\sin t \\ 0 \end{pmatrix} + \frac{r^2-h^2}{r^2} \cdot \frac{1}{r} \begin{pmatrix} \sqrt{r^2-h^2}\cos t \\ \sqrt{r^2-h^2}\sin t \\ h \end{pmatrix} \tag{360}$$

$$= \begin{pmatrix} -\dfrac{h^2\sqrt{r^2-h^2}}{r^2}\cos t \\ -\dfrac{h^2\sqrt{r^2-h^2}}{r^2}\sin t \\ \dfrac{h\left(r^2-h^2\right)}{r^2} \end{pmatrix} \tag{361}$$

また,$h=0$ のとき,$\dfrac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t) = 0$ で,$h \neq 0$ のとき,$\dfrac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t) \neq 0$ より,$h=0$ のとき,$\dot{\gamma}$ は γ に沿って平行であり,$h \neq 0$ のとき,$\dot{\gamma}$ は γ に沿って平行ではない

9.2 E10.2

(1)

$\forall t \in \mathbb{R}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \gamma(t) = \begin{pmatrix} a\cos t \\ a\sin t \\ kt \end{pmatrix} \tag{362}$$

とすると

$$x\sin\frac{z}{k} - y\cos\frac{z}{k} = a\cos t\sin t - a\sin t\cos t = 0 \tag{363}$$

よって,$\gamma(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S$.従って,$\gamma$ は S 内の曲線である

(2)

E2.1 の (2) より,S の単位法ベクトル場は

$$\mathbf{n}(\mathbf{p}) = \frac{1}{\sqrt{x^2+y^2+k^2}} \begin{pmatrix} k\sin\dfrac{z}{k} \\ -k\cos\dfrac{z}{k} \\ x\cos\dfrac{z}{k} + y\sin\dfrac{z}{k} \end{pmatrix} \tag{364}$$

である．また

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = \begin{pmatrix} -a\sin t \\ a\cos t \\ k \end{pmatrix} \tag{365}$$

$$\left\|\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)\right\| = \sqrt{a^2+k^2} \tag{366}$$

$$\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}(t) = \begin{pmatrix} -a\cos t \\ -a\sin t \\ 0 \end{pmatrix} \tag{367}$$

$$\mathbf{n}(\gamma(t)) = \frac{1}{\sqrt{a^2+k^2}}\begin{pmatrix} k\sin t \\ -k\cos t \\ a \end{pmatrix} \tag{368}$$

$$\det\begin{pmatrix} \frac{\mathrm{d}\gamma}{\mathrm{d}t} & \frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}(t) & \mathbf{n}(\gamma(t)) \end{pmatrix} = \frac{1}{\sqrt{a^2+k^2}}\begin{vmatrix} -a\sin t & -a\cos t & k\sin t \\ a\cos t & -a\sin t & -k\cos t \\ k & 0 & a \end{vmatrix} \tag{369}$$

$$= a\sqrt{a^2+k^2} \tag{370}$$

従って

$$\kappa_g(t) = \frac{1}{\left\|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\|^3}\det\begin{pmatrix} \frac{\mathrm{d}\gamma}{\mathrm{d}t} & \frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}(t) & \mathbf{n}(\gamma(t)) \end{pmatrix} \tag{371}$$

$$= \frac{a}{a^2+k^2} \tag{372}$$

また，$\left\|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\| = \sqrt{a^2+k^2}$ は定数だから，$\kappa_g \equiv 0 \iff a=0$．よって，$a=0$ のとき，γ は測地線で，$a\neq 0$ のとき，γ は測地線ではない

9.3 P10.1

(1)

$$X(t)\cdot\mathbf{n}(\gamma(t)) = \begin{pmatrix} -rt\sin t \\ rt\cos t \\ ht \end{pmatrix}\cdot\begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \tag{373}$$

$$= -rt\sin t\cos t + rt\cos t\sin t = 0 \tag{374}$$

であるから，$X(t)\in T_{\gamma(t)}S$ で，$X(t)$ は γ に沿う接ベクトル場である．また

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \begin{pmatrix} -r(\sin t + t\cos t) \\ r(\cos t - t\sin t) \\ h \end{pmatrix} \tag{375}$$

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t)\cdot\mathbf{n}(\gamma(t)) = -rt \tag{376}$$

$$\frac{\mathrm{D}X}{\mathrm{d}t}(t) = \left(\frac{\mathrm{d}X}{\mathrm{d}t}(t)\right)_{\tan} = \frac{\mathrm{d}X}{\mathrm{d}t} - \left(\frac{\mathrm{d}X}{\mathrm{d}t}(t)\cdot\mathbf{n}(\gamma(t))\right)\mathbf{n}(\gamma(t)) \tag{377}$$

$$= \begin{pmatrix} -r(\sin t + t\cos t) \\ r(\cos t - t\sin t) \\ h \end{pmatrix} + rt\begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} = \begin{pmatrix} -r\sin t \\ r\cos t \\ h \end{pmatrix} \tag{378}$$

第一成分と第二成分は常に 0 にならないので，$\dfrac{\mathrm{D}X}{\mathrm{d}t}(t) \not\equiv 0$．従って，$X$ は γ に沿って平行ではない

(2)

$$\dot{\gamma}(t) = \begin{pmatrix} -r\sin t \\ r\cos t \\ h \end{pmatrix} \tag{379}$$

$$\ddot{\gamma}(t) = \begin{pmatrix} -r\cos t \\ -r\sin t \\ 0 \end{pmatrix} \tag{380}$$

$$\mathbf{n}(\gamma(t)) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \tag{381}$$

$$\kappa_g(t) = \frac{1}{\|\dot{\gamma}\|^3} \det \begin{pmatrix} \dot{\gamma} & \ddot{\gamma} & \mathbf{n}(\gamma) \end{pmatrix} \tag{382}$$

$$= \frac{1}{(r^2+h^2)^{\frac{3}{2}}} \begin{vmatrix} -r\sin t & -r\cos t & \cos t \\ r\cos t & -r\sin t & \sin t \\ h & 0 & 0 \end{vmatrix} \tag{383}$$

$$= 0 \tag{384}$$

(3)

$$\dot{c} = \begin{pmatrix} -2rt\sin t^2 \\ 2rt\cos t^2 \\ 2ht \end{pmatrix}, \ddot{c} = \begin{pmatrix} -2r\left(\sin t^2 + 2t^2\cos t^2\right) \\ 2r\left(\cos t^2 - 2t^2\sin t^2\right) \\ 2h \end{pmatrix}, \mathbf{n}(c(t)) = \begin{pmatrix} \cos t^2 \\ \sin t^2 \\ 0 \end{pmatrix}$$

$$\|\dot{c}\| = \sqrt{4r^2t^2 + 4h^2t^2} = 2t\sqrt{r^2+h^2} \tag{385}$$

$$\kappa_g(t) = \frac{1}{\|\dot{c}\|^3} \det \begin{pmatrix} \dot{c} & \ddot{c} & \mathbf{n}(c) \end{pmatrix} \tag{386}$$

$$= \frac{1}{\left(2t\sqrt{r^2+h^2}\right)^3} \begin{vmatrix} -2rt\sin t^2 & -2r\left(\sin t^2 + 2t^2\cos t^2\right) & \cos t^2 \\ 2rt\cos t^2 & 2r\left(\cos t^2 - 2t^2\sin t^2\right) & \sin t^2 \\ 2ht & 2h & 0 \end{vmatrix} \tag{387}$$

$$= 0 \tag{388}$$

ここで，$\kappa_g(t) \equiv 0$ であるが，$t \neq 0$ なら $\|\dot{c}\| \neq Const$ なので，$\|\dot{c}\|$ は定数にならない．従って，c は測地線ではない

10 §12

10.1 E12.1

$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ を $g_{\mathbf{q}}(\xi, \eta)$ に代入すると

$$g(\mathbf{e}_1, \mathbf{e}_1) = \left(1 + u^2\right) \tag{389}$$
$$g(\mathbf{e}_1, \mathbf{e}_2) = uv \tag{390}$$
$$g(\mathbf{e}_2, \mathbf{e}_2) = \left(1 + v^2\right) \tag{391}$$

(1)

$$\epsilon_1 = \frac{\mathbf{e}_1}{\|\mathbf{e}_1\|} = \frac{\mathbf{e}_1}{\sqrt{g(\mathbf{e}_1, \mathbf{e}_1)}} = \frac{1}{\sqrt{1+u^2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{392}$$

$\mathbf{e}_2' := \mathbf{e}_2 - g(\mathbf{e}_2, \epsilon_1)\,\epsilon_1$ とおくと

$$\mathbf{e}_2' = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{1+u^2}} \\ 0 \end{pmatrix}\right) \cdot \frac{1}{\sqrt{1+u^2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{393}$$
$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{uv}{\sqrt{1+u^2}} \cdot \frac{1}{\sqrt{1+u^2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{394}$$
$$= \begin{pmatrix} -\dfrac{uv}{1+u^2} \\ 1 \end{pmatrix} \tag{395}$$
$$\|\mathbf{e}_2'\|_g^2 = g\left(\mathbf{e}_2', \mathbf{e}_2'\right) \tag{396}$$
$$= \frac{u^2v^2}{1+u^2} - \frac{2u^2v^2}{1+u^2} + \left(1+v^2\right) = 1 + v^2 - \frac{u^2v^2}{1+u^2} \tag{397}$$
$$= \frac{u^2v^2 + u^2 + v^2 + 1 - u^2v^2}{1+u^2} = \frac{u^2+v^2+1}{1+u^2} \tag{398}$$
$$\epsilon_2 = \frac{\mathbf{e}_{2'}}{\|\mathbf{e}_2'\|_g} \tag{399}$$
$$= \sqrt{\frac{1+u^2}{u^2+v^2+1}} \begin{pmatrix} -\dfrac{uv}{1+u^2} \\ 1 \end{pmatrix} \tag{400}$$

$$\begin{cases} \gamma_1 = \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \\ \gamma_2 = \begin{pmatrix} 1 \\ t-1 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \\ \gamma_3 = \begin{pmatrix} 3-t \\ 3-t \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \end{cases} \Longrightarrow \begin{cases} \dot{\gamma}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \dot{\gamma}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \dot{\gamma}_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \end{cases}$$

$\gamma_1, \gamma_2, \gamma_3$ をそれぞれ ϵ_1, ϵ_2 に代入すると

$$\epsilon_1 = \begin{pmatrix} \frac{1}{\sqrt{1+t^2}} \\ 0 \end{pmatrix} \qquad \epsilon_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \qquad \epsilon_1 = \begin{pmatrix} \frac{1}{\sqrt{t^2-6t+10}} \\ 0 \end{pmatrix}$$

$$\epsilon_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \epsilon_2 = \begin{pmatrix} \frac{1-t}{\sqrt{2t^2-4t+6}} \\ \frac{\sqrt{2}}{\sqrt{t^2-2t+3}} \end{pmatrix} \quad \epsilon_2 = \begin{pmatrix} \frac{(t-3)^2\sqrt{\frac{t^2-6t+10}{2t^2-12t+19}}}{t^2-6t+10} \\ \sqrt{\frac{t^2-6t+10}{2t^2-12t+19}} \end{pmatrix}$$

γ_1 に対して

$$\frac{\xi}{\|\xi\|_g} = \frac{\dot{\gamma}_1}{\|\dot{\gamma}_1\|_g} = \frac{1}{\sqrt{1+t^2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{401}$$

$$= \cos(\phi_1(t))\begin{pmatrix} \frac{1}{\sqrt{1+t^2}} \\ 0 \end{pmatrix} + \sin(\phi_1(t))\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{402}$$

から，$\begin{cases} \cos(\phi(t)) = 1 \\ \sin(\phi(t)) = 0 \end{cases}$ で，$\phi_1(0) = \phi_1(1) = 0$

γ_2 に対して

$$\frac{\xi}{\|\xi\|_g} = \frac{\dot{\gamma}_2}{\|\dot{\gamma}_2\|_g} = \frac{1}{\sqrt{t^2-2t+2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{403}$$

$$= \cos(\phi_2(t))\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} + \sin(\phi_2(t))\begin{pmatrix} \frac{1-t}{\sqrt{2t^2-4t+6}} \\ \frac{\sqrt{2}}{\sqrt{t^2-2t+3}} \end{pmatrix} \tag{404}$$

で，$\phi_2(1), \phi_2(2)$ はそれぞれ $\begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cos(\phi_2(1))\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} + \sin(\phi_2(1))\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \cos(\phi_2(2))\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} + \sin(\phi_2(2))\begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix} \end{cases}$

であるから，$\begin{cases} \phi_2(1) = \frac{\pi}{2} \\ \phi_2(2) = \frac{\pi}{3} \end{cases}$

γ_3 に対して

$$\frac{\xi}{\|\xi\|_g} = \frac{\dot{\gamma}_3}{\|\dot{\gamma}_3\|_g} = \frac{1}{\sqrt{(u+v)^2+2}}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{4t^2-24t+38}}\begin{pmatrix} -1 \\ -1 \end{pmatrix} \tag{405}$$

$$= \cos(\phi_3(t))\begin{pmatrix} \frac{1}{\sqrt{t^2-6t+10}} \\ 0 \end{pmatrix} + \sin(\phi_3(t))\begin{pmatrix} \frac{(t-3)^2\sqrt{\frac{t^2-6t+10}{2t^2-12t+19}}}{t^2-6t+10} \\ \sqrt{\frac{t^2-6t+10}{2t^2-12t+19}} \end{pmatrix} \tag{406}$$

で，$\phi_3(2), \phi_3(3)$ はそれぞれ

$$\begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} = \cos(\phi_3(2)) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} + \sin(\phi_3(2)) \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix} \tag{407}$$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \cos(\phi_3(3)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\phi_3(3)) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{408}$$

であるから，$\begin{cases} \phi_3(2) = \frac{7\pi}{6} \\ \phi_3(3) = \frac{5\pi}{4} \end{cases}$

(2)

$$\begin{cases} \phi_2(1) - \phi_1(1) - \theta_1 \in 2\pi\mathbb{Z} \\ \phi_3(2) - \phi_2(2) - \theta_2 \in 2\pi\mathbb{Z} \\ \phi_1(0) - \phi_3(3) - \theta_3 \in 2\pi\mathbb{Z} \end{cases} \Longrightarrow \begin{cases} \frac{\pi}{2} - 0 - \theta_1 \in 2\pi\mathbb{Z} \\ \frac{7\pi}{6} - \frac{\pi}{3} - \theta_2 \in 2\pi\mathbb{Z} \\ 0 - \frac{5\pi}{4} - \theta_3 \in 2\pi\mathbb{Z} \end{cases} \Longrightarrow \begin{cases} \theta_1 = \frac{\pi}{2} \\ \theta_2 = \frac{5\pi}{6} \\ \theta_3 = \frac{3\pi}{4} \end{cases}$$

(3)

$$\begin{aligned} \mathrm{Rot}_g(\gamma) &= \sum_{i=1}^{3} (\phi_i(t_i) - \phi_i(t_{i-1})) + \sum_{i=1}^{3} \theta_i && (409) \\ &= \left(0 - \frac{\pi}{6} + \frac{\pi}{12}\right) + \left(\frac{\pi}{2} + \frac{5\pi}{6} + \frac{3\pi}{4}\right) && (410) \\ &= 2\pi && (411) \end{aligned}$$

10.2 P12.1

$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ を $g_{\mathbf{q}}(\xi, \eta)$ に代入すると

$$g(\mathbf{e}_1, \mathbf{e}_1) = \frac{4}{(1 - u^2 - v^2)^2} \tag{412}$$

$$g(\mathbf{e}_1, \mathbf{e}_2) = 0 \tag{413}$$

$$g(\mathbf{e}_2, \mathbf{e}_2) = \frac{4}{(1 - u^2 - v^2)^2} \tag{414}$$

(1)

$$\epsilon_1 = \frac{\mathbf{e}_1}{\|\mathbf{e}_1\|_g} = \frac{\mathbf{e}_1}{\sqrt{g(\mathbf{e}_1, \mathbf{e}_1)}} = \frac{1 - u^2 - v^2}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{415}$$

$\mathbf{e}'_2 := \mathbf{e}_2 - g\left(\mathbf{e}_2, \epsilon_1\right)\epsilon_1$ とおくと

$$\mathbf{e}'_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1-u^2-v^2}{2} \\ 0 \end{pmatrix}\right)\begin{pmatrix} \frac{1-u^2-v^2}{2} \\ 0 \end{pmatrix} \tag{416}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{417}$$

$$\epsilon_2 = \frac{\mathbf{e}_{2'}}{\|\mathbf{e}'_2\|_g} \tag{418}$$

$$= \frac{\left(1 - u^2 - v^2\right)}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{419}$$

$$\begin{cases} \gamma_1 = \begin{pmatrix} a\cos t \\ a\sin t \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \\ \gamma_2 = \begin{pmatrix} \frac{a}{2}\left(\cos t - 1\right) \\ \frac{a}{2}\sin t \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \\ \gamma_3 = \begin{pmatrix} -\frac{a}{2}\left(\cos t - 1\right) \\ -\frac{a}{2}\sin t \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \end{cases} \implies \begin{cases} \dot{\gamma}_1 = \begin{pmatrix} -a\sin t \\ a\cos t \end{pmatrix} \\ \dot{\gamma}_2 = \begin{pmatrix} -\frac{a}{2}\sin t \\ \frac{a}{2}\cos t \end{pmatrix} \\ \dot{\gamma}_3 = \begin{pmatrix} \frac{a}{2}\sin t \\ -\frac{a}{2}\cos t \end{pmatrix} \end{cases}$$

ここで，$\lambda := \dfrac{1 - u^2 - v^2}{2}$ とし，$\gamma_1, \gamma_2, \gamma_3$ をそれぞれ代入すると

$$\lambda_1 = \frac{1 - a^2}{2} \tag{420}$$

$$\lambda_2 = \frac{a^2\cos t - a^2 + 2}{4} \tag{421}$$

$$\lambda_3 = \frac{a^2\cos t - a^2 + 2}{4} \tag{422}$$

$\gamma_1, \gamma_2, \gamma_3$ をそれぞれ ϵ_1, ϵ_2 に代入すると

$$\begin{matrix} \epsilon_1 = \lambda_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} & \epsilon_1 = \lambda_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} & \epsilon_1 = \lambda_3\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \epsilon_2 = \lambda_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} & \epsilon_2 = \lambda_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} & \epsilon_2 = \lambda_3\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{matrix}$$

から，λ を代入すると

$$\begin{matrix} \epsilon_1 = \begin{pmatrix} \frac{1-a^2}{2} \\ 0 \end{pmatrix} & \epsilon_1 = \begin{pmatrix} \frac{a^2\cos t - a^2 + 2}{4} \\ 0 \end{pmatrix} & \epsilon_1 = \begin{pmatrix} \frac{a^2\cos t - a^2 + 2}{4} \\ 0 \end{pmatrix} \\ \epsilon_2 = \begin{pmatrix} 0 \\ \frac{1-a^2}{2} \end{pmatrix} & \epsilon_2 = \begin{pmatrix} 0 \\ \frac{a^2\cos t - a^2 + 2}{4} \end{pmatrix} & \epsilon_2 = \begin{pmatrix} 0 \\ \frac{a^2\cos t - a^2 + 2}{4} \end{pmatrix} \end{matrix}$$

γ_1 に対して

$$\frac{\xi}{\|\xi\|_g} = \frac{\dot{\gamma}_1}{\|\dot{\gamma}_1\|_g} = \frac{1 - u^2 - v^2}{2a}\begin{pmatrix} -a\sin t \\ a\cos t \end{pmatrix} = \frac{1 - a^2}{2}\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \tag{423}$$

$$= \cos\left(\phi_1\left(t\right)\right)\begin{pmatrix} \dfrac{1 - a^2}{2} \\ 0 \end{pmatrix} + \sin\left(\phi_1\left(t\right)\right)\begin{pmatrix} 0 \\ \dfrac{1 - a^2}{2} \end{pmatrix} \tag{424}$$

で

$$\begin{pmatrix} 0 \\ \dfrac{1-a^2}{2} \end{pmatrix} = \cos\left(\phi_1\left(0\right)\right)\begin{pmatrix} \dfrac{1-a^2}{2} \\ 0 \end{pmatrix} + \sin\left(\phi_1\left(0\right)\right)\begin{pmatrix} 0 \\ \dfrac{1-a^2}{2} \end{pmatrix} \tag{425}$$

$$\begin{pmatrix} 0 \\ -\dfrac{1-a^2}{2} \end{pmatrix} = \cos\left(\phi_1\left(\pi\right)\right)\begin{pmatrix} \dfrac{1-a^2}{2} \\ 0 \end{pmatrix} + \sin\left(\phi_1\left(\pi\right)\right)\begin{pmatrix} 0 \\ \dfrac{1-a^2}{2} \end{pmatrix} \tag{426}$$

より，$\begin{cases} \phi_1\left(0\right) = \dfrac{\pi}{2} \\ \phi_1\left(\pi\right) = \dfrac{3\pi}{2} \end{cases}$

γ_2 に対して

$$\frac{\xi}{\left\|\xi\right\|_g} = \frac{\dot{\gamma}_2}{\left\|\dot{\gamma}_2\right\|_g} = \frac{1-u^2-v^2}{2}\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = \frac{a^2\cos t - a^2 + 2}{4}\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \tag{427}$$

$$= \cos\left(\phi_2\left(t\right)\right)\begin{pmatrix} \dfrac{a^2\cos t - a^2 + 2}{4} \\ 0 \end{pmatrix} + \sin\left(\phi_2\left(t\right)\right)\begin{pmatrix} 0 \\ \dfrac{a^2\cos t - a^2 + 2}{4} \end{pmatrix} \tag{428}$$

で

$$\begin{pmatrix} 0 \\ \dfrac{a^2-1}{2} \end{pmatrix} = \cos\left(\phi_2\left(\pi\right)\right)\begin{pmatrix} \dfrac{1-a^2}{2} \\ 0 \end{pmatrix} + \sin\left(\phi_2\left(\pi\right)\right)\begin{pmatrix} 0 \\ \dfrac{1-a^2}{2} \end{pmatrix} \tag{429}$$

$$\begin{pmatrix} 0 \\ -\dfrac{1}{2} \end{pmatrix} = \cos\left(\phi_2\left(2\pi\right)\right)\begin{pmatrix} \dfrac{1}{2} \\ 0 \end{pmatrix} + \sin\left(\phi_2\left(2\pi\right)\right)\begin{pmatrix} 0 \\ \dfrac{1}{2} \end{pmatrix} \tag{430}$$

より，$\begin{cases} \phi_2\left(\pi\right) = \dfrac{3\pi}{2} \\ \phi_2\left(2\pi\right) = \dfrac{\pi}{2} \end{cases}$

γ_3 に対して

$$\frac{\xi}{\left\|\xi\right\|_g} = \frac{\dot{\gamma}_3}{\left\|\dot{\gamma}_3\right\|_g} = \frac{1-u^2-v^2}{2}\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} = \frac{a^2\cos t - a^2 + 2}{4}\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} \tag{431}$$

$$= \cos\left(\phi_3\left(t\right)\right)\begin{pmatrix} \dfrac{a^2\cos t - a^2 + 2}{4} \\ 0 \end{pmatrix} + \sin\left(\phi_3\left(t\right)\right)\begin{pmatrix} 0 \\ \dfrac{a^2\cos t - a^2 + 2}{4} \end{pmatrix} \tag{432}$$

で

$$\begin{pmatrix} 0 \\ -\dfrac{1}{2} \end{pmatrix} = \cos\left(\phi_3\left(2\pi\right)\right)\begin{pmatrix} \dfrac{1}{2} \\ 0 \end{pmatrix} + \sin\left(\phi_3\left(2\pi\right)\right)\begin{pmatrix} 0 \\ \dfrac{1}{2} \end{pmatrix} \tag{433}$$

$$\begin{pmatrix} 0 \\ \dfrac{1-a^2}{2} \end{pmatrix} = \cos\left(\phi_3\left(3\pi\right)\right)\begin{pmatrix} \dfrac{1-a^2}{2} \\ 0 \end{pmatrix} + \sin\left(\phi_3\left(3\pi\right)\right)\begin{pmatrix} 0 \\ \dfrac{1-a^2}{2} \end{pmatrix} \tag{434}$$

より，$\begin{cases} \phi_3\left(2\pi\right) = \dfrac{3\pi}{2} \\ \phi_3\left(3\pi\right) = \dfrac{\pi}{2} \end{cases}$

(2)

$$\begin{cases}\phi_2(\pi)-\phi_1(\pi)-\theta_1\in 2\pi\mathbb{Z}\\ \phi_3(2\pi)-\phi_2(2\pi)-\theta_2\in 2\pi\mathbb{Z}\\ \phi_1(0)-\phi_3(3\pi)-\theta_3\in 2\pi\mathbb{Z}\end{cases} \implies \begin{cases}\dfrac{3\pi}{2}-\dfrac{3\pi}{2}-\theta_1\in 2\pi\mathbb{Z}\\ \dfrac{3\pi}{2}-\dfrac{\pi}{2}-\theta_2\in 2\pi\mathbb{Z}\\ \dfrac{\pi}{2}-\dfrac{\pi}{2}-\theta_3\in 2\pi\mathbb{Z}\end{cases} \implies \begin{cases}\theta_1=0\\ \theta_2=-\pi\\ \theta_3=0\end{cases}$$

(3)

$$\mathrm{Rot}_g(\gamma)=\sum_{i=1}^{3}\left(\phi_i(t_i)-\phi_i(t_{i-1})\right)+\sum_{i=1}^{3}\theta_i \tag{435}$$

$$=\left(\frac{3\pi}{2}-\frac{\pi}{2}\right)+\left(\frac{\pi}{2}-\frac{3\pi}{2}\right)+\left(\frac{\pi}{2}-\frac{3\pi}{2}\right)+0-\pi+0 \tag{436}$$

$$=2\pi \tag{437}$$

11　§13

11.1　E13.1

(1)

$$\angle A+\angle B+\angle C=\iint_{\triangle ABC}K\mathrm{d}A+\pi \tag{438}$$

$$=\iint_{\triangle ABC}\frac{1}{r^2}\mathrm{d}A+\pi \tag{439}$$

$$>\pi \tag{440}$$

(2)

$x\cdot y=0$ をみたす $\mathbf{x},\mathbf{y}\in S^2\left(r\right)$ に対し，$\gamma\left(t\right)=x\cos t+y\sin t$ は $\mathbf{x},\mathbf{y}$ を通る測地線で，その弧長パラメータ表示 $\gamma\left(s\right)=x\cos\frac{s}{r}+y\sin\frac{s}{r}$ も測地線

1. $\mathbf{a}\cdot\mathbf{b}=\begin{pmatrix}r\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\r\\0\end{pmatrix}=0$ より，$\gamma_1:\left[0,\frac{\pi r}{2}\right]\to S^2\left(r\right)$ を $\gamma_1\left(s\right)=\mathbf{a}\cos\frac{s}{r}+\mathbf{b}\sin\frac{s}{r}=\begin{pmatrix}r\cos\frac{s}{r}\\r\sin\frac{s}{r}\\0\end{pmatrix}$ で定めると，$\gamma_1\left(0\right)=\begin{pmatrix}r\\0\\0\end{pmatrix},\gamma_2\left(\frac{\pi r}{2}\right)=\begin{pmatrix}0\\r\\0\end{pmatrix}$ なので，γ_1 が $\mathbf{a}$ を始点 $\mathbf{b}$ を終点とする弧長パラメータ表示された最短の $S^2\left(r\right)$ 内の測地線
2. $\mathbf{a}\cdot\mathbf{c}=\begin{pmatrix}r\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\r\end{pmatrix}=0$ より，$\gamma_2:\left[0,\frac{\pi r}{2}\right]\to S^2\left(r\right)$ を $\gamma_2\left(s\right)=\mathbf{a}\cos\frac{s}{r}+\mathbf{c}\sin\frac{s}{r}=\begin{pmatrix}r\cos\frac{s}{r}\\0\\r\sin\frac{s}{r}\end{pmatrix}$ で定める，$\gamma_2\left(0\right)=\begin{pmatrix}r\\0\\0\end{pmatrix},\gamma_2\left(\frac{\pi r}{2}\right)=\begin{pmatrix}\frac{r}{\sqrt{2}}\\0\\\frac{r}{\sqrt{2}}\end{pmatrix}$ なので，γ_2 は $\mathbf{a}$ を始点 $\mathbf{c}$ を終点とする弧長パラメータ表示された最短の $S^2\left(r\right)$ 内の測地線

(3)

$\gamma_1'\left(s\right)=\begin{pmatrix}-\sin\frac{s}{r}\\\cos\frac{s}{r}\\0\end{pmatrix},\gamma_2'\left(s\right)=\begin{pmatrix}-\sin\frac{s}{r}\\0\\\cos\frac{s}{r}\end{pmatrix}$ より，$\gamma_1'\left(0\right)=\begin{pmatrix}0\\1\\0\end{pmatrix},\gamma_2'\left(0\right)=\begin{pmatrix}0\\0\\1\end{pmatrix}$ である．よって

$$\frac{\gamma_1'\left(0\right)\cdot\gamma_2'\left(0\right)}{\|\gamma_1'\left(0\right)\|\,\|\gamma_2'\left(0\right)\|}=0 \tag{441}$$

となるので，$\angle A=\frac{\pi}{2}$

(4)

$\angle B,\angle C$ も同様に計算すると，$\angle B=\frac{\pi}{4},\angle C=\frac{\pi}{2}$ となる．ガウス・ボンネの定理より

$$\frac{1}{r^2}\iint_{\triangle ABC}\mathrm{d}A=\angle A+\angle B+\angle C-\pi \tag{442}$$

$$=\frac{\pi}{4} \tag{443}$$

よって，$Area\left(\triangle ABC\right)=\dfrac{\pi}{4}r^2$

11.2 P13.1

(1)

$\hat{a}=\dfrac{\mathbf{a}}{r},\hat{b}=\dfrac{\mathbf{b}}{r},\hat{c}=\dfrac{\mathbf{c}}{r}$ とおく

1. $\hat{a}\cdot\hat{b}=\begin{pmatrix}\frac{1}{2}\\0\\\frac{\sqrt{3}}{2}\end{pmatrix}\cdot\begin{pmatrix}0\\-\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix}=\dfrac{3}{4},\dfrac{\hat{a}\cdot\hat{b}}{\|\hat{a}\|\left\|\hat{b}\right\|}=\arccos\dfrac{3}{4}$ で，$\epsilon_1=\hat{a}$ とし

$$\epsilon_2=\frac{\hat{b}-\left(\hat{b}\cdot\hat{a}\right)\hat{a}}{\left\|\hat{b}-\left(\hat{b}\cdot\hat{a}\right)\hat{a}\right\|} \tag{444}$$

$$=\frac{4}{\sqrt{7}}\begin{pmatrix}-\frac{3}{8}\\-\frac{1}{2}\\\frac{3\sqrt{3}}{8}\end{pmatrix}=\begin{pmatrix}-\frac{3}{2\sqrt{7}}\\-\frac{2}{\sqrt{7}}\\\frac{3\sqrt{3}}{2\sqrt{7}}\end{pmatrix} \tag{445}$$

$$\text{よって,}\gamma_{ab}\left(s\right)=r\left(\cos\left(\frac{s}{r}\begin{bmatrix}\frac{1}{2}\\0\\\frac{\sqrt{3}}{2}\end{bmatrix}\right)+\sin\left(\frac{s}{r}\begin{bmatrix}-\frac{3}{2\sqrt{7}}\\-\frac{2}{\sqrt{7}}\\\frac{3\sqrt{3}}{2\sqrt{7}}\end{bmatrix}\right)\right)\qquad 0\le s\le r\arccos\frac{3}{4}$$

2. $\hat{b}\cdot\hat{c}=\begin{pmatrix}0\\-\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\end{pmatrix}=\dfrac{1}{2\sqrt{3}}-\dfrac{1}{2},\dfrac{\hat{b}\cdot\hat{c}}{\left\|\hat{b}\right\|\|\hat{c}\|}=\arccos\left(\dfrac{1}{2\sqrt{3}}-\dfrac{1}{2}\right)$ で，$\epsilon_1=\hat{b}$ とし

$$\epsilon_2=\frac{\hat{c}-\left(\hat{c}\cdot\hat{b}\right)\hat{b}}{\left\|\hat{c}-\left(\hat{c}\cdot\hat{b}\right)\hat{b}\right\|} \tag{446}$$

$$=\frac{\sqrt{6}}{\sqrt{4+\sqrt{3}}}\begin{pmatrix}\frac{1}{\sqrt{3}}\\-\frac{1+\sqrt{3}}{4}\\-\frac{3+\sqrt{3}}{12}\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{2}}{\sqrt{4+\sqrt{3}}}\\-\frac{3+\sqrt{3}}{4\left(4+\sqrt{3}\right)}\\-\frac{3+\sqrt{3}}{2\sqrt{6\left(4+\sqrt{3}\right)}}\end{pmatrix} \tag{447}$$

$$\text{よって，}\gamma_{bc}\left(s\right)=r\left(\cos\left(\frac{s}{r}\begin{bmatrix}0\\-\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{bmatrix}\right)+\sin\left(\frac{s}{r}\begin{bmatrix}\frac{\sqrt{2}}{\sqrt{4+\sqrt{3}}}\\-\frac{3+\sqrt{3}}{4\left(4+\sqrt{3}\right)}\\-\frac{3+\sqrt{3}}{2\sqrt{6\left(4+\sqrt{3}\right)}}\end{bmatrix}\right)\right)\text{ で，ここで,}$$

$$0 \leq s \leq r \arccos\left(\frac{1}{2\sqrt{3}} - \frac{1}{2}\right)$$

3. $\hat{c} \cdot \hat{a} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{1}{2\sqrt{3}} - \frac{1}{2}, \frac{\hat{c} \cdot \hat{a}}{\|\hat{c}\| \|\hat{a}\|} = \arccos\left(\frac{1}{2\sqrt{3}} - \frac{1}{2}\right)$ で，$\epsilon_1 = \hat{c}$ とし

$$\epsilon_2 = \frac{\hat{a} - (\hat{a} \cdot \hat{c})\,\hat{c}}{\|\hat{a} - (\hat{a} \cdot \hat{c})\,\hat{c}\|} \tag{448}$$

$$= \frac{\sqrt{6}}{\sqrt{4+\sqrt{3}}} \begin{pmatrix} \frac{2+\sqrt{3}}{6} \\ \frac{1-\sqrt{3}}{6} \\ \frac{1}{6} + \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{2+\sqrt{3}}{\sqrt{6\left(4+\sqrt{3}\right)}} \\ \frac{1-\sqrt{3}}{\sqrt{6\left(4+\sqrt{3}\right)}} \\ \frac{1+2\sqrt{3}}{\sqrt{6\left(4+\sqrt{3}\right)}} \end{pmatrix} \tag{449}$$

よって，$\gamma_{ca}(s) = r \left(\cos\left(\frac{s}{r} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} \right) + \sin\left(\frac{s}{r} \begin{bmatrix} \frac{2+\sqrt{3}}{\sqrt{6\left(4+\sqrt{3}\right)}} \\ \frac{1-\sqrt{3}}{\sqrt{6\left(4+\sqrt{3}\right)}} \\ \frac{1+2\sqrt{3}}{\sqrt{6\left(4+\sqrt{3}\right)}} \end{bmatrix} \right) \right)$ で，ここで，

$$0 \leq s \leq r \arccos\left(\frac{1}{2\sqrt{3}} - \frac{1}{2}\right)$$

(2)

(1) の計算より，

$$\begin{cases} v_{a\to b} = \hat{b} - \left(\hat{b} \cdot \hat{a}\right)\hat{a} = \begin{pmatrix} -\frac{3}{8} \\ -\frac{1}{2} \\ \frac{3\sqrt{3}}{8} \end{pmatrix} & \|v_{a\to b}\| = \frac{\sqrt{7}}{4} \\ v_{a\to c} = \hat{c} - (\hat{c} \cdot \hat{a})\,\hat{a} = \begin{pmatrix} \frac{1+\sqrt{3}}{4} \\ -\frac{1}{\sqrt{3}} \\ -\frac{3+\sqrt{3}}{12} \end{pmatrix} & \|v_{a\to c}\| = \frac{\sqrt{4+\sqrt{3}}}{\sqrt{6}} \end{cases}$$

$$\begin{cases} v_{b\to c} = \hat{c} - \left(\hat{c}\cdot\hat{b}\right)\hat{b} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1+\sqrt{3}}{4} \\ -\frac{3+\sqrt{3}}{12} \end{pmatrix} & \|v_{b\to c}\| = \frac{\sqrt{4+\sqrt{3}}}{\sqrt{6}} \\ v_{b\to a} = \hat{a} - \left(\hat{a}\cdot\hat{b}\right)\hat{b} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{8} \\ \frac{\sqrt{3}}{8} \end{pmatrix} & \|v_{b\to a}\| = \frac{\sqrt{7}}{4} \end{cases}$$

$$\begin{cases} v_{c\to a} = \hat{a} - \left(\hat{a}\cdot\hat{c}\right)\hat{c} = \begin{pmatrix} \frac{2+\sqrt{3}}{6} \\ \frac{1-\sqrt{3}}{6} \\ \frac{1}{6}+\frac{1}{\sqrt{3}} \end{pmatrix} & \|v_{c\to a}\| = \frac{\sqrt{4+\sqrt{3}}}{\sqrt{6}} \\ v_{c\to b} = \hat{b} - \left(\hat{b}\cdot\hat{c}\right)\hat{c} = \begin{pmatrix} \frac{-1+\sqrt{3}}{6} \\ \frac{-2-\sqrt{3}}{6} \\ \frac{2}{\sqrt{3}}-\frac{1}{6} \end{pmatrix} & \|v_{c\to b}\| = \frac{\sqrt{10-\sqrt{3}}}{\sqrt{6}} \end{cases}$$

であるから

$$\cos\angle A = \frac{v_{a\to b}\cdot v_{a\to c}}{\|v_{a\to b}\|\,\|v_{a\to c}\|} = -\frac{9+\sqrt{3}}{2\sqrt{42\left(4+\sqrt{3}\right)}} \tag{450}$$

$$\cos\angle B = \frac{v_{b\to a}\cdot v_{b\to c}}{\|v_{b\to a}\|\,\|v_{b\to c}\|} = \frac{-3+\sqrt{3}}{\sqrt{42\left(4+\sqrt{3}\right)}} \tag{451}$$

$$\cos\angle C = \frac{v_{c\to a}\cdot v_{c\to b}}{\|v_{c\to a}\|\,\|v_{c\to b}\|} = \frac{25+4\sqrt{3}}{6\sqrt{37+6\sqrt{3}}} \tag{452}$$

よって，
$$\begin{cases} \angle A = \arccos\left(-\frac{9+\sqrt{3}}{2\sqrt{42\left(4+\sqrt{3}\right)}}\right) \\ \angle B = \arccos\left(\frac{-3+\sqrt{3}}{\sqrt{42\left(4+\sqrt{3}\right)}}\right) \\ \angle C = \arccos\left(\frac{25+4\sqrt{3}}{6\sqrt{37+6\sqrt{3}}}\right) \end{cases}$$

(3)

$$Area\left(\triangle ABC\right) = r^2\left(\angle A + \angle B + \angle C - \pi\right) \tag{453}$$

$$= r^2 \left(\arccos\left(-\frac{9+\sqrt{3}}{2\sqrt{42\left(4+\sqrt{3}\right)}} \right) + \arccos\left(\frac{-3+\sqrt{3}}{\sqrt{42\left(4+\sqrt{3}\right)}} \right) + \arccos\left(\frac{25+4\sqrt{3}}{6\sqrt{37+6\sqrt{3}}} \right) - \pi \right) \tag{454}$$

12 §14

12.1 E14.1

$f : S^2(1) \to S, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \frac{r}{\left(x^{2m}+y^{2m}+z^{2m}\right)^{\frac{1}{2m}}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ は同相写像で，実際，$g : S \to S^2(1)$ が f の逆写像で，f と g は連続であるから，S と $S^2(1)$ は同相．よって，$\chi(S) = \chi\left(S^2(1)\right) = 2$

12.2 E14.2

前の計算より，$T_{R,r}$ のガウス曲率 K は $K = \frac{1}{r^2}\left(1 - \frac{R}{\sqrt{x^2+y^2}}\right)$ となる．また，ガウス・ボンネの定理より

$$2\pi\chi(T_{R,r}) = \iint_{T_{R,r}} K \mathrm{d}A \tag{455}$$

$$= \frac{1}{r^2}\iint_{T_{R,r}} \left(1 - \frac{R}{\sqrt{x^2+y^2}}\right) \mathrm{d}A \tag{456}$$

ここで，$T_{R,r}$ の局所パラメータ表示 $\sigma : (0,2\pi)^2 \to \mathbb{R}^3$ を $\sigma(u,v) = \begin{pmatrix} (R+r\cos u)\cos v \\ (R+r\cos u)\sin v \\ r\sin u \end{pmatrix}$ すると，$(0,2\pi)^2$ は面積確定な有界集合．$T_{R,r}\backslash\sigma\left((0,2\pi)^2\right)$ の面積は 0 で，さらに $\|\sigma_u \times \sigma_v\| = r(R+r\cos u)$ なので

$$\int_{T_{R,r}} \left(1 - \frac{R}{\sqrt{x^2+y^2}}\right) \mathrm{d}A = \iint_{(0,2\pi)^2} \left(1 - \frac{R}{R+r\cos u}\right) r(R+r\cos u)\,\mathrm{d}u\mathrm{d}v \tag{457}$$

$$= r^2 \int_0^{2\pi} \left(\int_0^{2\pi} \cos u \mathrm{d}v\right) \mathrm{d}u \tag{458}$$

$$= 2\pi r^2 \int_0^{2\pi} \cos u \mathrm{d}u \tag{459}$$

$$= 0 \tag{460}$$

よって，$\chi(T_{R,r}) = \frac{1}{2\pi}\iint_{T_{R,r}} \left(1 - \frac{R}{\sqrt{x^2+y^2}}\right) \mathrm{d}A = 0$

12.3 E14.3

(1)

E を $\triangle_1, \cdots, \triangle_{24}$ の辺全体の集合，$F = \{\triangle_1, \cdots, \triangle_{24}\}$ とする．このとき，$\#V = v, \#F = 24, \#E = \frac{3}{2}\cdot\#F = 36$．よって，$\chi(S) = \#V - \#E + \#F = v - 12$

(2)

(1) より，$v = 10$ とすると，$\chi(S) = -2$ となり，ガウス・ボンネの定理より，$\iint_S K \mathrm{d}A = 2\pi\chi(S) = -4\pi < 0$．ここで，$\forall \mathbf{p} \in S, K \geq 0$ と仮定すると，$\iint_S K \mathrm{d}A \geq 0$ となるので，矛盾．よって，$\exists \mathbf{p} \in S, s.t. K(\mathbf{p}) < 0$

12.4 P14.1

$$\chi\left(S^2(r)\right) = \frac{1}{2\pi}\iint_{S^2(r)} K\mathrm{d}A \tag{461}$$

$$= \frac{1}{2\pi}\iint_{S^2(r)} \frac{1}{r^2}\mathrm{d}A \tag{462}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{r^2}\cdot 4\pi r^2 \tag{463}$$

$$= 2 \tag{464}$$

12.5 P14.2

$$\chi(S) = \frac{1}{2\pi}\iint_S K\mathrm{d}A \tag{465}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^L \left(-\frac{\ddot{\rho}(s)}{\rho(s)}\right)\cdot\rho(s)\,\mathrm{d}s\mathrm{d}\theta \tag{466}$$

$$= \frac{1}{2\pi}\cdot 2\pi\int_0^L (-\ddot{\rho}(s))\,\mathrm{d}s \tag{467}$$

$$= -\left[\dot{\rho}(s)\right]_0^L \tag{468}$$

ここで，$\rho \in S$ は閉曲線であるから

$$\chi(S) = -\left[\dot{\rho}(s)\right]_0^L = 0 \tag{469}$$